INTEGRABILITY OF TRIGONOMETRIC SERIES AND TRANSFORMS

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we give a fairly extensive bibliography of various publications which have been referred to in the present thesis.

It may be mentioned here that the major portion of this thesis, presented in the form of papers, has been communicated for publication in various international mathematical journals. Two of them have already been accepted for publication in " Proceedings of the American Mathematical Society " and " Publicationes Mathematicae 20 (1973) "

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C O N T E N T S

Chapter I

INTRODUCTION

1.1 The Theory of Integrability of trigonometric series and transforms is of recent origin. It was in 1949 that the celebrated mathematician Professor B. Sz-Nagy, a distinguished researcher in the domain of functional analysis, initiated the discussion of problems concerning this topic. Within a short period of two decades, it has now grown into a fully developed discipline of mathematical analysis. The present thesis is based on certain investigations of the author into the theory of "Integrability of Trigonometric Series and Transforms."

1.2 Before giving a résumé of the earlier researches, in the light of which various new and interesting results have been obtained by the author, it seems desirable to state here the definitions and notations which will be required in the sequel.

ORLICZ AND YOUNG FUNCTIONS

A non-decreasing, continuous, real-valued function Φ defined on the non-negative half-line and vanishing only at the origin will be called an Orlicz function (OF). Function $\Phi \in$ OF is said to satisfy $\triangle_\alpha$ ($\alpha > 0$) condition for large u if there are constants $c > 0$ and $u_o \geq 0$ such that $\Phi(\alpha u) \leq C\ \Phi(u)$, $u \geq u_o$ for every $\alpha > 1$. A convex Orlicz function Φ satisfying the conditions

$$\lim_{u \to o} \frac{\Phi(u)}{u} = 0 \quad \text{and} \quad \lim_{u \to \infty} \frac{\Phi(u)}{u} = \infty$$

is called a Young function (YF). Function Φ belongs to YF if, and only if, it admits a representation

$$\Phi(u) = \int_o^u \phi(x)\, dx ,$$

where $\phi(x)$, $x \geq 0$, is positive, $\phi(o) = 0$, continuous on the right, non-decreasing and $\lim_{x \to \infty} \phi(x) = \infty$.

We have for such functions the relation

$$\frac{\Phi(u)}{u} \leq \phi(u) \leq \frac{\Phi(2u)}{u} .$$

We denote by M the class of Orlicz functions Φ which satisfy the following condition of Mulholland [1].

" There exist a convex function Λ, $\lambda > 1$ and $0 < k < 1$, such that the inequality $\Lambda(u) \leq \Phi^k(u) \leq \lambda \Lambda(u)$ holds for all u."

Let $L_\Phi(X, \mu)$, where $\Phi \in \triangle_\alpha$, be the Orlicz space, i.e., the set of all complex valued measurable functions f on a measure space (X, μ) such that the modular $\int_X \Phi(|f(x)|)\, d\mu$ is finite. By Hardy-Orlicz space H_Φ we mean simply closed subspace of $L_\Phi((0, 2\pi), dx)$ spanned over trigonometric polynomials of the form

$$f(t) = \sum_{n=0}^{N} a_n e^{int} .$$

<u>Functions of $L(p,\alpha)$ class</u>.

A function $\phi(x)$ is said to belong to the class $L(p,\alpha)$ (Askey and Wainger [1]) if

$$\int_0^\pi |\phi(x)|^p \sin^{\alpha p} x \, dx < \infty ,$$

where α is any real number and $p > 0$.

We define the norm of a function $\phi(x) \in L(p,\alpha)$ as :

$$\| \phi(x) \|_{p,\alpha} = \left\{ \int_0^\pi |\phi(x)|^p \sin^{\alpha p} x \, dx \right\}^{1/p} .$$

It is evident that $L(p,\alpha) \Rightarrow L^p$ for $\alpha < 0$, $L^p \Rightarrow L(p,\alpha)$ for $\alpha > 0$ and $L^p \equiv L(p,\alpha)$ when $\alpha = 0$.

<u>Slowly increasing function</u>.

A positive function $L(x)$ is said to be " Slowly increasing in the sense of Karamata [1] if it is continuous for $x \geq 0$ and

$$\lim_{x\to\infty} \frac{L(tx)}{L(x)} = 1 \qquad \text{for every } t > 0.$$

Monotonic sequences.

A sequence $\{a_n\}$ of non-negative numbers is said to be quasi-monotone (Shah [1] ; Szász [1]) if for some $\alpha > 0$,

$$a_{n+1} \leq a_n \left(1 + \frac{\alpha}{n} \right)$$

for all $n > n_o(\alpha)$, where $n_o(\alpha)$ is a positive number depending upon α.

An equivalent definition of quasi-monotone sequence (Shah [2]) is that $n^{-\beta} a_n \downarrow 0$ for some $\beta > 0$.

It is said to be quasi-monotone of $\alpha = \alpha_o$ (Yong [2]) if

$$a_{n+1} \leq a_n \left(1 + \frac{\alpha_o}{n} \right).$$

A sequence $\{a_n\}$ is said to be δ-quasi-monotone (Boas [2]) if $a_n \to 0$, $a_n > 0$ ultimately and $\Delta a_n \geq -\delta_n$, where $\{\delta_n\}$ is a sequence of positive numbers.

Fourier series and transforms.

Let $f(x)$ and $g(x)$ be defined by the following trigonometric series.

(1.2.1) $$f(x) = \frac{1}{2} a_o + \sum_{n=1}^{\infty} a_n \cos nx ,$$

(1.2.2) $$g(x) = \sum_{n=1}^{\infty} a_n \sin nx .$$

Let $F(x)$ be the Fourier sine transforms of $f(t)$, that is to say

$$F(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin xt \, dt.$$

1.3 <u>Integrability of trigonometric series of L-class with monotonic coefficients.</u>

Concerning the integrability of trigonometric series for monotonic coefficients, Boas [1] , in 1952, proved the following theorems[†]

<u>Theorem A.</u> If $a_n \downarrow 0$ and $0 < \gamma < 1$, then $x^{-\gamma} f(x) \in L(0,\pi)$ if, and only if $\sum\limits_{n=1}^{\infty} n^{\gamma-1} a_n < \infty$.

<u>Theorem B.</u> If $a_n \downarrow 0$ and $0 \leq \gamma \leq 1$, then $x^{-\gamma} g(x) \in L(0,\pi)$ if, and only, if $\sum\limits_{n=1}^{\infty} n^{\gamma-1} a_n < \infty$.

Later on Sunouchi [1] proved these theorems by using a different method.

In 1962, Shah [2] extended above theorems in the following form.

<u>Theorem C.</u> Let $\{a_n\}$ be a quasi-monotone.

(i) If $0 < \gamma < 1$, then $\sum\limits_{n=1}^{\infty} n^{\gamma-1} a_n$ is convergent if, and only if, $x^{-\gamma} f(x) \in L(0,\pi)$.

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† For detailed literature concerning L-class reference may be made to the recent monograph of Boas [3] .

(ii) If $0 < \gamma \leq 1$, then $\sum_{n=1}^{\infty} n^{\gamma-1} a_n$ is convergent if, and only if, $x^{-\gamma} g(x) \in L(0,\pi)$.

Theorem C was extended by Boas [2] for δ-quasi-monotonic sequences in the following manner.

<u>Theorem D</u>. Let $0 < \gamma < 1$, and let $\{a_n\}$ be δ-quasi-monotonic sequence with $\sum_{n=1}^{\infty} n^{\gamma} \delta_n < \infty$, then $\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ converges (except perhaps at integral multiples of 2π) to $f(x)$, and $\sum_{n=1}^{\infty} n^{\gamma-1} a_n$ converges if, and only if, $x^{-\gamma} f(x) \in L(0,\pi)$.

<u>Theorem E</u>. Let $0 < \gamma \leq 1$, and let $\{a_n\}$ be δ-quasi-monotonic sequence with $\sum_{k=1}^{\infty} n^{\gamma} \delta_n < \infty$, then $\sum_{n=1}^{\infty} a_n \sin nx$ converges to $g(x)$ and $\sum_{n=1}^{\infty} n^{\gamma-1} a_n$ converges if, and only if, $x^{-\gamma} g(x) \in L(0,\pi)$.*

Note: Suppose that $\{a_n\}$ is a positive sequence tending to zero and $\triangle a_n > -\frac{\alpha a_n}{n}$. Let $\sum_{n=1}^{\infty} n^{\gamma-1} a_n < \infty$ and $0 < \gamma < 1$. Then Theorem D asserts that $\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ converges (except perhaps at integral multiples of 2π) to $f(x)$ and $\sum_{n=1}^{\infty} n^{\gamma-1} a_n$ converges if and only, $x^{-\gamma} f(x) \in L(0,\pi)$. Considering

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*Theorem E ($\Sigma \to f$) part has been generalized in a different direction by Hasegawa [2] .

the case $\int - \Sigma$ we observe that Boas had already assumed what he wished to prove. Thus his results (Theorem D and E) suffer from such a defect. Of course, he could have avoided this by (say in Theorem D) assuming the convergence of $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ instead of that of $\sum n^{\gamma} \delta_n$ in part $\int - \Sigma$.

Aljančić, Bojanić and Tomić [2], in 1955, generalized Theorem A and B in a different direction. They proved, among others, the following results.

<u>Theorem F</u>. If $0 < \gamma < 1$, $a_n \downarrow 0$, then $x^{-\gamma} L(1/x) f(x) \in L(0,\pi)$, if and only if, $\sum_{n=1}^{\infty} n^{\gamma-1} L(n) a_n < \infty$.

<u>Theorem G</u>. If $0 < \gamma < 2$, $a_n \downarrow 0$, then $x^{-\gamma} L(1/x)\ g(x) \in L(0,\pi)$, if and only if, $\sum_{n=1}^{\infty} n^{\gamma-1} L(n)\ a_n < \infty$.

These results were subsequently extended by Yong [1], in 1965, to quasi-monotonic sequences in the following form.

<u>Theorem H</u>. Let $\{a_n\}$ be a quasi-monotone sequence with $a_n \to 0$ as $n \to \infty$ and $0 < \gamma < 1$. Then $\sum_{n=1}^{\infty} n^{\gamma-1} L(n)\ a_n$ converges if, and only if, $\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ converges everywhere to $f(x)$, save possibly $x=0$, and $x^{-\gamma} L(1/x)\ f(x) \in L(0,\pi)$.

<u>Theorem I</u>. Let $\{a_n\}$ be a quasi-monotone sequence with $a_n \to 0$, as $n \to \infty$.

(i) For $0 < \gamma < p$, if $\sum_{n=1}^{\infty} n^{\gamma-1} L(n)\, a_n$ converges, then $\sum_{n=1}^{\infty} a_n \sin nx$ converges everywhere to $g(x)$ and $x^{-\gamma} L(1/x)\, g(x) \in L(0,\pi)$.

(ii) For $0 < \gamma < 1$, if $\sum_{n=1}^{\infty} a_n \sin nx$ converges everywhere to $g(x)$ and $x^{-\gamma} L(1/x)\, g(x) \in L(0,\pi)$, then $\sum_{n=1}^{\infty} n^{\gamma-1} L(n) a_n$ converges.

It may be remarked that if we examine the proof of $\int \to \Sigma$ in Theorem H and I we find that it is sufficient to assume that $\{a_n\}$ is only a positive sequence.

The Chapter V of this thesis is concerned with the generalization of all the results stated above. We prove the following theorems.

<u>Theorem 1</u>. <u>Let</u> $\{a_n\}$ <u>be a</u> δ-<u>quasi-monotone sequence and</u> $0 < \gamma < 1$. <u>If</u>

$$(1.3.1) \qquad \sum_{n=1}^{\infty} n^{\gamma} L(n)\, \delta_n < \infty,$$

<u>and</u>

$$(1.3.2) \qquad \sum_{n=1}^{\infty} n^{\gamma-1} L(n)\, a_n$$

<u>converges, then</u>

$$(1.3.3) \qquad \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

converges, everywhere to $f(x)$ except possibly at $x = 0$ and

$$(1.3.4) \qquad x^{-\gamma} L(^1/x) f(x) \in L(0,\pi).$$

Conversely, if $\{a_n\}$ is any sequence which is ultimately positive such that (1.3.3) holds and if (1.3.4) holds, then (1.3.2) holds.

Theorem 2. (i) Let $\{a_n\}$ be a δ-quasi-monotone sequence and $0 < \gamma < 2$. If $\sum_{n=1}^{\infty} n^{\gamma} L(n)\, \delta_n$ and $\sum_{n=1}^{\infty} n^{\gamma-1} L(n)\, a_n$ are convergent, then $\sum_{n=1}^{\infty} a_n \sin nx$ converges everywhere to $g(x)$ and $x^{-\gamma} L(^1/x)\, g(x) \in L(0,\pi)$.

(ii) For $0 < \gamma < 1$, if $\sum_{n=1}^{\infty} a_n \sin nx$ converges everywhere to $g(x)$ and $x^{-\gamma} L(^1/x)\, g(x) \in L(0,\pi)$, then $\sum_{1}^{\infty} n^{\gamma-1} L(n)\, a_n$ converges where $\{a_n\}$ is any sequence which is ultimately positive.

1.4 Recently, Hasegawa [1] has proved the following Theorems.

Theorem J. Let $f(x)$ be an even function, continuous on $(0,\pi)$, and let its Fourier series be

$$(1.4.1) \qquad f(x) \sim \frac{1}{2} a_0 + \sum_1^\infty a_n \cos nx .$$

Suppose that $\alpha(x)$ is an even function, positive and non-increasing on $(0,\pi)$, that $x\,\alpha(x) \in L(0,\pi)$ and that there is a positive number $\eta \leq \pi$ such that

$$t^{-2} \int_0^t x\,\alpha(x)\,dx \leq C\,\alpha(t) \qquad \text{for all } t, \quad 0 < t \leq \eta.$$

If the series $\sum_{n=1}^\infty a_n \int_{1/n}^\pi \alpha(x)\,dx$ converges absolutely, then the Fourier series (1.4.1) converges uniformly to $f(x)$ and $(\,f(x) - f(s)\,)\,\alpha\,(x-s) \in L(0,\pi)$ for each s, $0 < s < \pi$.

<u>Theorem K</u>. Let $g(x)$ be an odd function, continuous on $(0,\pi)$, and let its Fourier series be

$$(1.4.2) \qquad g(x) \sim \sum_1^\infty b_n \sin nx .$$

Suppose that $\alpha(x)$ is an odd function, positive on $(0,\pi)$, that $x\alpha(x)$ is Lebesgue integrable and non-increasing on $(0,\pi)$ and that there is a positive number $\eta \leq \pi$ such that

$$t^{-1} \int_t^\eta \alpha(x)\,dx \leq C\,\alpha(t)$$

for all t, $0 < t \leq \eta$. If the series $\sum_{n=1}^\infty n\,b_n \int_0^{1/n} x\,\alpha(x)\,dx$

converges absolutely, then the Fourier series (1.4.2) converges uniformly to $g(x)$ and $(\,g(x) - g(s)\,)\,\alpha(x-s) \in L(0,\pi)$ for each

s, $0 \le s \le \pi$. In particular, in case $s=0$, Parseval's formula

$$\frac{2}{\pi}\int_0^\pi g(x)\ \alpha(x)\ dx = \sum_{n=1}^{\infty} b_n\, q_n$$

holds, where q_n's are defined by

$$q_n = \frac{2}{\pi}\int_0^\pi \alpha(x)\ \sin nx\ dx\ ,$$

and the series $\sum_{n=1}^{\infty} b_n\, q_n$ converges absolutely.

In Chapter VII, we have obtained certain generalizations of Theorems J and K. For example Theorem J has been generalized in the following form.

In what follows we assume that $\alpha(x)$, $\phi(x)$ and $\psi(x)$ are positive functions defined on $(0,\pi)$, such that $\alpha(x)\ \phi(x) \in L(0,\pi)$ and $\alpha(x)\ \psi(x)$ is either even or odd.

<u>Theorem 3. Let</u> $f(x)$ <u>be an even function and continuous on</u> $(0,\pi)$ <u>and let its Fourier series be</u> (1.4.1). <u>Suppose that</u> $\alpha(x)\ \dfrac{\psi(x)}{\phi(x)}$ <u>is non-increasing on</u> $(0,\pi)$ <u>and that there is a positive number</u> $\eta \le \pi$ <u>such that</u>

$$\int_0^t \phi(x)\ \alpha(x)\ dx \ \le\ C\ \Phi_1(t)\ \alpha(t)$$

<u>for all</u> t, $0 < t \le \eta$, <u>where</u> $\Phi_1(t) = \int_0^t \phi(x)\ dx$. <u>If the</u>

series $\sum_{n=1}^{\infty} a_n \int_{1/n}^{\pi} \alpha(x)\,\psi(x)\,dx$ *converges absolutely, then the Fourier series* (1.4.1) *converges uniformly to* $f(x)$ *and* $(f(x) - f(s))\,\alpha(x-s)\,\psi(x-s) \in L(0,\pi)$ *for each* s, $0 < s < \pi$, *where* $x\,\dfrac{\psi(x)}{\phi(x)} < M$, $0 < x \leq \pi$ *and*

$$\frac{\phi(x)\;\Phi_1(x)}{\psi(x)\,(\,\Phi_1(x) - \Phi_1(\frac{x}{2}))} = O(x)\,, \quad x \to 0.$$

By taking $\psi(x) = 1$ and $\phi(x) = x$ we get theorem J. On the other hand if we take $\phi(x) = x$ and $\psi(x)\,\alpha(x) = \beta(x)$ in Theorem 3, we get Theorem J in which one of our conditions namely, " $\frac{\beta(x)}{x} \downarrow$ " is lighter than the corresponding condition "$\beta(x) \downarrow$" in Theorem J. Also we extend the scope of our theorem by assuming that $\beta(x)$ is either even or odd instead of restricting it to be even in Theorem J. Thus our Theorem is a definite generalization of Theorem J.

1.5 *Integrability of trigonometric series of L^p and other associated classes with monotone coefficients.*

Concerning the integrability of trigonometric series for L^p class, Hardy and Littlewood (see Zygmund [1]) proved the following theorems.

Theorem L. If $a_n \downarrow 0$ and $1 < p < \infty$, then $f \in L^p(0,\pi)$

if and only if, $\sum_{n=1}^{\infty} n^{p-2} a_n^p < \infty$, where $f(x) \sim \frac{1}{2}a_0 + \sum_1^{\infty} a_n \cos nx$. Similar theorem holds for sine series.

They also obtained the following dual theorem in which decreasing coefficients were replaced by a decreasing function.

Theorem B. If $f \geq 0$ and decreases, $1 < p < \infty$, and a_n are the Fourier cosine coefficents of f, then $\sum_1^{\infty} |a_n|^p < \infty$, if, and only if, $x^{p-2} f^p \in L(0,\pi)$.

A similar result is true for sine series.

In 1956, Theorem A was extended by Chen [1] in the following form.

Theorem C. Suppose that $a_n \downarrow 0$, $f(x) = \frac{a_0}{2} + \sum_1^{\infty} a_n \cos nx$. Then for $p > 1$, $0 < \gamma < 1$, $x^{-\gamma} f^p(x) \in L(0,\pi)$ if and only , $\sum_1^{\infty} n^{\gamma+p-2} a_n^p < \infty$. The result is also true for sine series.

He has also shown that if a_n is ultimately positive and decreases steadily to zero as $n \to \infty$, then Theorem A remains true.

Later on he [3] extended his results to other values of γ. His results were subsequently extended by Robertson [1] for the case when $\{a_n\}$ can be partitioned into k monotonic sequences.

A different kind of generalization has been discussed by Chen([2] ; [3]) in great details in a series of papers which generalize not only the power function multipliers but also the L^p classes.

Izumi [1] , in 1960, obtained a theorem as an extension of Theorem A in a different direction. This theorem was subsequently generalized by Yong [2] for quasi-monotone sequences. His theorem is as follows :

<u>Theorem C</u>. Let $\{a_n\}$ be quasi-monotone of $\alpha < 1$ and such that $C_2 \geq n^{\beta} L_1(n)\, a_n \geq C_1 > 0$ with some $\beta > 0$ (n=1,2,...). If $p \geq 1$ and $1 > \gamma > 1-p$, then $x^{-\gamma} L_2(1/x)\, f^p(x)$ is integrable $L(0,\pi)$ if, and only if, $\sum_1^\infty n^{\gamma+p-2} L_2(n)\, a_n^p$ converges, where $L_1(x)$ $L_2(x)$ are " slowly increasing " functions in the sense of Karamata and $f(x) = \sum_1^\infty a_n \cos nx$.

Similar result holds for sine series for $1+p > \gamma > 1-p$.

Mikarev and Nesaev [1] have obtained a theorem for Lorenz space $L(q,p)$*. Concerning the space $L(p,\alpha)$ Askey and Wainger [1

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*We say that a function $\phi \in L(q,p)$ $(1<p<\infty,\ 1<q<\infty)$ if$(t^{1/q-1/p}\phi^*)$ $\in L^p$, where ϕ^* is ϕ rearranged in decreasing order, i.e. ϕ^* is the decreasing function equimeasurable with ϕ. Similarly $\{a_n\} \in \ell(q,p)$ if$\{n^{\frac{1}{q}-\frac{1}{p}} a_n\} \in \ell^p$.

in 1966, have established the following theorems.

<u>Theorem P</u>. Let $f(x) \in L(p,\alpha)$ with $1 \leq p < \infty$, $-1 < \alpha p \leq p-1$. Let $f(x) \sim \sum_{1}^{\infty} a_n \cos nx$ with $a_n \geq 0$ and $A_n = \sum_{j=[\frac{n}{2}]}^{n} a_j$, then $\sum_{n=1}^{\infty} n^{-2-\alpha p} A_n^p < \infty$, and $\sum_{n=1}^{\infty} n^{-2-\alpha p} A_n^p \leq C(\alpha,p) \| f \|_{p,\alpha}^p$.

<u>Theorem Q</u>. Let $1 \leq p < \infty$, $-1 < \alpha p < p-1$. Suppose that $\{a_n\}$ is a sequence of numbers such that $a_n \to 0$ and $\left\{ \sum_{n=1}^{\infty} n^{p-\alpha p-2} \left(\sum_{k=n}^{\infty} |\triangle a_k| \right)^p \right\}^{1/p} < \infty$, then $f(x) \sim \sum_{1}^{\infty} a_n \cos nx$ is in $L(p,\alpha)$ class and $\| f \|_{p,\alpha}^p \leq B(\alpha,p) \sum_{n=1}^{\infty} n^{p-\alpha p-2} \left(\sum_{k=n}^{\infty} |\triangle a_k| \right)^p$,

where $\triangle a_k = a_k - a_{k+1}$.

From theorems P and Q they deduced the following interesting result.

<u>Theorem R</u>. Let $\{a_n\}$ be a positive sequence tending to zero and $\{n^{-k} a_n\}$ be monotonic decreasing for some non-negative integer k. Let $1 \leq p < \infty$ and $-1 < \alpha p < p-1$, then a necessary and sufficient condition that $f(x) \in L(p,\alpha)$ is that $\sum_{1}^{\infty} n^{p-\alpha p-2} a_n^p < \infty$, where $f(x) \sim \sum_{1}^{\infty} a_n \cos nx$.

Later on Khan [1] in 1968, obtained several results involving $L(p,\alpha)$ class, which generalize all the above results for cosine series. One of his typical result is as follows.

Theorem B. Let $\{a_n\}$ be a positive sequence tending to zero and $\{n^{-k} a_n\}$ be monotonic decreasing for some non-negative integer k. If $1 \leq p < \infty$, and $-1 < \alpha p < p-1$, then $L^{1/p}(\frac{1}{x})\ f(x) \in L(p,\alpha)$ if, and only if, $\sum_1^\infty n^{p-\alpha p-2} L(n)\ a_n^p < \infty$, where $f(x) \sim \sum_1^\infty a_n \cos nx$.

In Chapter IV of the present thesis we have obtained certain generalization of all these theorems. A number of results have been obtained. Some of the interesting results are as follows .

Theorem 4. Let $\lambda(1/x)\ L^{1/p}(\frac{1}{x})\ f(x) \in L(p,\alpha)$, with $1 \leq p < \infty$, $-1 < \alpha p < p-1$, where $f(x) \sim \sum_1^\infty a_n \cos nx$, with $a_n \geq 0$. If $A_n = \sum_{j=[\frac{n}{\rho}]}^{n} a_j$, then $\sum_1^\infty n^{-\rho-\alpha p} L(n)\ \lambda^p(n)\ A_n^p < \infty$, and

$$\sum_1^\infty n^{-2-\alpha p}\ L(n) \lambda^p(n) A_n^p \leq B(\alpha,p) \| L^{1/p}(1/x)\lambda(\pi/x)\ f(x) \|_{p,\alpha}^p ,$$

where $\lambda(x)$ is a positive function such that $(i)\ x^{-\alpha+1-1/p+\delta}\ \lambda(x) \uparrow$ as $x \to \infty$ for some small $\delta > 0$.

Theorem 5. Let $1 \leq p < \infty$ and $-1 < \alpha p < p-1$. Suppose that $\{a_n\}$ is a sequence of numbers such that $a_n \to 0$ and

$$\left\{ \sum_{1}^{\infty} n^{p-\alpha p-2} \lambda^{p}(n)\, L(n) \left(\sum_{j=n}^{\infty} |\Delta a_j| \right)^{p} \right\}^{1/p} < \infty . \quad \text{Then}$$

$$L^{1/p}(1/x)\ \lambda(1/x)\ f(x) \in L(p,\alpha),$$

and

$$\| \lambda(1/x) L^{1/p}(1/x) f(x) \|_{p,\alpha}^{p} \le B(\alpha,p) \sum_{n=1}^{\infty} n^{p-\alpha p-2} \lambda^{p}(n) L(n) \left(\sum_{j=n}^{\infty} |\Delta a_j| \right)^{p}$$

where $\lambda(x)$ *is a positive function such that* (i) $x^{-\alpha-1/p+\delta}\lambda(x) \downarrow$ as $x \to \infty$ for some small $\delta > 0$, and $f(x) \sim \sum_{1}^{\infty} a_n \cos nx$.

Combining Theorems 4 and 5 we deduce the following.

Theorem 6. *Let* $\{a_n\}$ *be a positive sequence tending to zero and* $\{n^{-k} a_n\}$ *be monotonic decreasing for some real number* k. *If* $1 \le p < \infty$, $-1 < \alpha p < p-1$ *and* $0 \le \gamma < \alpha + 1/p$, *then* $x^{-\gamma} L^{1/p}(1/x)\ f(x) \in L(p,\alpha)$ *if, and only if,*

$$\sum_{n=1}^{\infty} n^{p-\alpha p+\gamma p-2} L(n)\ a_n^{p} < \infty, \text{ where } f(x) \sim \sum_{1}^{\infty} a_n \cos nx .$$

1.6 *Integrability of trigonometric series of* L^p *class with positive coefficients.*

In the previous section we studied integrability problems of L^p and other associated classes with monotonic coefficients. In this connection R.P. Boas raised the question as to what would happen if the condition of monotonicity on the coefficients a_n is replaced by the condition of its non-negativeness. R. Askey gave an example to demonstrate that the condition $\sum_{1}^{\infty} n^{p\gamma+p-2}\, a_n^{p} < \infty$

for $p > \frac{2}{1+\gamma}$ is atleast not necessary and sufficient condition for the function $x^{-\gamma} f(x) \in L^p$. However, Boas [4] found a necessary and sufficient condition for such a case in a different form. He proved the following.

Theorem T. If $a_n \geq 0$ and a_n are the Fourier sine or cosine coefficients of a function $\phi(x)$, and $1<p<\infty$, $\frac{1}{p}<\gamma<\frac{1+p}{p}$, then

$$|x-a|^{-\gamma} |\phi(x)-\phi(a)| \in L^p, \quad 0 \leq a < \pi,$$

if, and only if,

$$\sum_{n=1}^{\infty} n^{p\gamma-p-2} \left(\sum_{k=1}^{n} k\, a_k\right)^p < \infty,$$

or equivalently

$$\sum_{n=1}^{\infty} n^{p\gamma-2} \left(\sum_{k=n}^{\infty} a_k\right)^p < \infty.$$

1.7 Integrability of trigonometric series of L^p and other associated classes with monotonic function

Concerning the function instead of its Fourier coefficients a_n, Askey and Boas [1] proved a number of theorems. One of their results is as follows.

Theorem U. Let $G(x) \downarrow$ on $(0,\pi)$, G bounded below and $\int_0^\pi x\, d\,G(x)$ finite so that dG has generalized sine coefficients $b_n = \frac{2}{\pi}\int_0^\pi \sin nx\, dG(x)$. If $1 < p < \infty$ and $\frac{1}{p} < \gamma < 1+\frac{1}{p}$, then $\{n^{-\gamma} b_n\} \in \ell^p$ if, and only if, $t^{\gamma-1-p/p} \int_0^t x\, dG(x) \in L^p$.

From their results, Askey and Boas deduced a number of interesting corollaries.

Recently Mazhar and Khan [1] have generalized all the main results of Askey and Boas [1]. Their generalization of Theorem U is as follows.

Theorem V. Let $G(x)$ satisfy the conditions of Theorem U. If $1 < p < \infty$ and $\lambda(x)$ is a positive function such that

(1.7.1) $x^{1+\delta}\lambda(x)$ is decreasing for some small $\delta > 0$,

(1.7.2) $x^{p+1-\delta}\lambda(x) \uparrow +\infty$ for some small $\delta > 0$, as $x \to \infty$,

then $\{\lambda^{1/p}(n)\, b_n\} \in \ell^p$, if, and only if, $\lambda^{1/p}(\frac{1}{t})\, t^{-1-2/p}\int_0^t x\,dG(x) \in L^p$.

In Chapter III of this thesis we have established four theorems which generalise various results of Mazhar and Khan [1]. Thus for example Theorem V has been generalized in the following form.

Theorem 7. Let $G(x)$ satisfy the conditions of Theorem U. If $1 < p < \infty$, $-1 < \alpha p < p-1$, then $\{n^{-\alpha}\lambda^{1/p}(n)b_n\} \in \ell^p$, if, and only if, $\lambda^{1/p}(\frac{1}{t})\, t^{-1-2/p}(\int_0^t x\,dG(x)) \in L(p,\alpha)$, where $\lambda(x)$ is a positive function such that

(i) $x^{1+\delta-\alpha p}\,\lambda(x)\downarrow$ <u>for some small</u> $\delta > 0$,

(ii) $x^{p+1-\delta}\,\lambda(x)\uparrow +\infty$ <u>for some small</u> $\delta > 0$, <u>as</u> $x \to \infty$.

Recently, M. Izumi and S. Izumi [1] have proved some theorems involving certain inequalities for Fourier series. With the help of some of their results they have obtained a theorem connecting monotonic decreasing Fourier coefficients with the corresponding generating function. We quote here one typical result. We write

$$G(x) = \int_{x/2}^{x} \frac{f(t)}{t}\,dt.$$

<u>Theorem A</u>. Let $p > 1$ and $a > -1$ and let f be a non-negative, non-increasing and integrable function on $(0,\pi)$. If $x^{a} f^{p}(x)$ is integrable, then we have

$$\int_0^{\pi} x^{a}\, G^{p}(x)\,dx \le A \int_0^{\pi} x^{a}\left(f(\tfrac{x}{2}) - f(x) \right)^{p} dx + A\left(\int_{\pi/2}^{\pi} f(x)\,dx\right)^{p}$$

where A is some positive constant.

In Chapter VI we have replaced the special class L^{p} of functions by a more general class L_{Φ} satisfying certain properties . Our results generalize all the theorems of M. Izumi and S. Izumi. Thus for example Theorem A has been generalized in the following manner.

<u>Theorem 8. Let</u> $\Phi \in \Delta_2 \cap YF$ <u>and</u> f <u>be a non-negative and integrable on</u> $(0,\pi)$. <u>If</u> $x^\beta \Phi(f(x))$ <u>is integrable and</u> $\beta > -1$, <u>then we have</u>

$$\int_0^\pi x^\beta \Phi(G(x))\, dx \le A \int_0^\pi x^\beta \Phi(|f(x)-f(\tfrac{x}{2})|)\,dx + A\, \Phi\Big(\int_{\pi/2}^\pi f(x)\,dx\Big).$$

<u>where A is some positive constant.</u>

1.8 <u>Integrability of power series.</u>

In 1955, Heywood [1] studied the integrability of power series and proved several theorems. His main theorem is as follows .

<u>Theorem K.</u> Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $a_n \ge 0$, $0 \le x < 1$.

Then for $\gamma < 1$,

$$(1-x)^{-\gamma} f(x) \in L(0,1)$$

if, and only if,

$$\sum_{n=1}^{\infty} n^{\gamma-1} a_n < \infty .$$

Later on he [2] weakened the hypothesis of positiveness of the coefficients a_n by replacing it with $a_n \ge -\frac{K}{n^{1+\epsilon}}$,

$C > 0$. He proved the following theorem.

<u>Theorem Y</u>. Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $0 \leq x < 1$, $\gamma < 1$ and that there is a positive number C such that $a_n > -\frac{K}{n^{1+C}}$, for all sufficiently large values of n, K being some positive constant. Then $(1-x)^{-\gamma} f(x) \in L(0,1)$ if, and only if,

$$\sum_{n=1}^{\infty} n^{\gamma-1} |a_n| < \infty .$$

In the same year, Kennedy [1] replaced $(1-x)^{-\gamma}$ by a general integrable function which contains Theorem Y for $0 \leq \gamma < 1$. He also proved a general theorem which contains Theorem Y for $\gamma < 0$. Some of the theorems of Heywood were subsequently generalised by Boas and González-Fernández [1]. Chen [4] made certain generalization in another direction.

Later on in 1964 Mercer [1] proved some theorems on integrability of power series involving Cauchy integrals.

Askey [1] and Askey and Boas [2] have proved several theorems on the integrability of power series with positive coefficients for L^p class. Recently, Khan [2] and Mazhar and Khan [2] generalized the results of Askey [1] and Askey and Boas [2] The result of Khan, which deals with the case $a_n \geq 0$, includes as a special case for p=1 the theorem of Heywood (Theorem X). Almost simultaneously, Leindler [1] generalized the result of Khan.

The question arises whether it is possible to relax the condition of positiveness of a_n in the theorems of Khan. In Section B of Chapter II we answer this question in the affirmative. We prove the following theorem which includes Theorem Y of Heywood for p=1.

<u>Theorem</u> 9. <u>Let</u> $f(x) = \sum_{0}^{\infty} a_n x^n$, $0 \leq x < 1$ <u>and</u> $\gamma < 1$. Suppose that there is a positive number ϵ such that $a_n > -\frac{K}{n^{\gamma/p+1-1/p+\epsilon}}$ $(0 < p \leq \infty)$ <u>for all sufficiently large values of</u> n, <u>where</u> K <u>is some positive constant</u>. <u>Then</u> $(1-x)^{-\gamma} (f(x))^p \in L(0,1)$ <u>if, and only if</u> $\Sigma n^{\gamma-2} (\sum_{k=1}^{n} |a_k|)^p$ <u>converges</u>.

Recently, Woyczyński [1] proved the following theorem which deals with the equivalence of a number of statements pertaining to integrability of power series.

<u>Theorem Z</u>. Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $0 \leq x < 1$. If $a_n \geq a_{n+1} \geq 0$ $(n = 0,1,2,\ldots)$ then the following four statements are equivalent.

(i) $f(x) \in L_{\Phi}(0, 1)$;

(ii) $g(t) \equiv f(e^{it}) \in H_{\Phi}(0,2\pi)$;

(iii) $\{n\, a_n\} \in L_{\Phi}(N, \nu)$;

(iv) $\{A_n\} \in L_{\Phi}(N, \nu)$,

where $\Phi \in \Delta_\alpha \cap E \cap YP$, $d\mu = dx$, N stands for the set of all positive integers and ν is the measure on N concentrating the mass n^{-2} at the point $n \in N$, and $A_n = a_0 + a_1 + \dots + a_n$.

Section A of Chapter II deals with a generalization of Theorem 2. We prove the following theorem.

<u>Theorem</u> 10. <u>Let</u> $\Phi \in \Delta_\alpha \cap E \cap YP$ <u>and</u>

$$f(x) = \sum_{k=0}^{\infty} a_k x^k, \quad 0 \leq x < 1.$$

<u>If</u> $\{a_n\}$ <u>is a quasi-monotone sequence such that</u> $0 < B_1 \leq n^{\beta} a_n \leq B_2$ <u>with some</u> $\beta > 0$, $(n=1,2,\dots)$, <u>and</u> $0 \leq \gamma < 1$, <u>then the following four statements are equivalent</u>.

(i) $(1-x)^{-\gamma} \Phi(f(x)) \in L(0,1)$;

(ii) $x^{-\gamma} \Phi(|f(e^{ix})|) \in L(0,\pi)$;

(iii) $\sum_{n=1}^{\infty} n^{\gamma-2} \Phi(n a_n) < \infty$;

(iv) $\sum_{n=1}^{\infty} n^{\gamma-2} \Phi(A_n) < \infty$,

<u>where</u> $A_n = a_0 + a_1 + \dots + a_n$.

1.9 <u>Integrability of trigonometric transforms</u>.

Recently Boas [6] has proved the following theorem for

Fourier transforms by a method which is rather more direct than those that have been used for similar problems about Fourier series.

<u>Theorem</u> A'. If $f(x) \downarrow 0$, $x^{1/p} f(x) \in L^p(0,1)$, and $F(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin xt\, dt$, then $x^{\gamma+1-2/p} F(x) \in L^p(0,\infty)$ provided that $x^{-\gamma} f(x) \in L^p(0,\infty)$, where $p > 1$ and $-\frac{1}{p} < \gamma < \frac{1}{p}$.

It may be remarked that in Theorem A', the condition $x^{1/p} f(x) \in L^p(0,1)$ is redundant, in view of the condition $x^{-\gamma} f(x) \in L^p(0,\infty)$.

In Chapter VIII of this thesis we obtain a theorem in which we have replaced L^p class by a more general class L_Φ. Our theorem is as follows.

<u>Theorem 11. Let $F(x)$ be a sine transform of $f(x)$. If $f(x) \downarrow 0$, $x^{-\alpha} \Phi(f(x)) \in L(0,\infty)$ and $-1 < \alpha < 1$, then $x^{\alpha-2} \Phi(x F(x)) \in L(0, \infty)$, where $\Phi(x)$ is a convex Orlicz function satisfying Δ_2 condition.</u>

It may be observed that for $\Phi(t) = t^p$, $p > 1$, we get Theorem A'.

Chapter II

ON THE INTEGRABILITY OF POWER SERIES*

2.1 A non-decreasing continuous real-valued function Φ defined on the non-negative half line and vanishing only at the origin will be called an Orlicz function (OF). Function $\Phi \in$ OF is said to satisfy Δ_α $(\alpha > 0)$ condition for large u if there are constants $C > 0$ and $u_0 \geq 0$ such that $\Phi(\alpha u) \leq C\, \Phi(u)$, $u \geq u_0$. A convex Orlicz function Φ satisfying the conditions

$$\lim_{u\to 0} \frac{\Phi(u)}{u} = 0 \quad \text{and} \quad \lim_{u\to\infty} \frac{\Phi(u)}{u} = \infty,$$

is called a Young function (YF). Function Φ belongs to YF iff it admits a representation of the form

$$\Phi(u) = \int_0^u \phi(t)\, dt,$$

where $\phi(t)$, $t \geq 0$, is positive, $\phi(0) = 0$, continuous on the right, non-decreasing and $\lim_{t\to\infty} \phi(t) = \infty$

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* Section A has been accepted for publication in Proceedings of American Mathematical Society, while Section B in Publicationes Mathematicae 20(1973).

We denote by M the class of Orlicz functions Φ which satisfy the following condition of Mulholland [1] . " There exist a convex function Λ , $\lambda > 1$ and $0 < \alpha < 1$, such that the inequality

$$\Lambda(u) \leq \Phi^{\alpha}(u) \leq \lambda \Lambda(u) \text{ holds for all } u."$$

A sequence $\{a_n\}$ of non-negative numbers is said to be quasi-monotone (Shah [1] ; Szasz [1]) if for some $\alpha > 0$,

$$a_{n+1} \leq a_n \left(1 + \frac{\alpha}{n} \right).$$

An equivalent definition of quasi-monotone sequence is that $n^{-\beta} a_n \downarrow 0$ for some $\beta > 0$ (Shah [2]).

Let $L_{\Phi}(X, \mu)$, where $\Phi \in \Delta_{\alpha}$, be the Orlicz space, i.e., the set of all complex valued measurable functions f on a measure space (X, μ) such that the modular $\int_X \Phi(|f(x)|)d\mu$ is finite. In this chapter by Hardy-Orlicz space H_{Φ} we mean simply a closed subspace of $L_{\Phi}((0,2\pi), dx)$ spanned over trigonometric polynomials of the form

$$f(t) = \sum_{n=0}^{N} a_n e^{int} .$$

This chapter consists of two section. In section A we obtain a generalisation of certain results concerning equivalence of some statements pertaining to integrability of a power series, while in section B, we study integrability problem for a power series under certain Tauberian condition.

SECTION A

2.2 Recently, Woyczyński [1] proved the following theorem.

Theorem A. Let

$$f(x) = \sum_{k=0}^{\infty} a_k x^k , \quad 0 \leq x < 1.$$

If $a_n \geq a_{n+1} \geq 0$ ($n = 0, 1, 2, ---$), then the following four statements are equivalent :

(2.2.1) $\quad f(x) \in L_{\Phi}(0, 1)$;

(2.2.2) $\quad g(t) \equiv f(e^{it}) \in H_{\Phi}(0, 2\pi)$;

(2.2.3) $\quad \{n a_n\} \in L_{\Phi}(N, \nu)$;

(2.2.4) $\quad \{A_n\} \in L_{\Phi}(N, \nu)$,

where $\Phi \in \Delta_{\alpha} \cap M \cap YF$, $d\mu = dx$, N stands for the set of all positive integers and ν is the measure on N concentrating the mass n^{-2} at the point $n \in N$, and $A_n = a_0+a_1+a_2+ \dots a_n$.

In this section our object is to obtain a generalisation of the above theorem. We denote throughout this section by

B (with or without suffixes) a positive constant, not necessarily the same at each occurrence.

We prove the following Theorem.

<u>Theorem 1. Let</u> $\Phi \in \Delta_\alpha \cap M \cap YF$ <u>and</u> $f(x) = \sum_{k=0}^{\infty} a_k x^k$,

$0 \leq x < 1$. <u>If</u> $\{a_n\}$ <u>is a quasi-monotone sequence such that</u> $0 < B_1 \leq n^\beta a_n \leq B_2$ <u>with some</u> $\beta > 0$, $(n = 1,2,\dots)$, <u>and</u> $0 \leq \gamma < 1$, <u>then the following four statements are equivalent</u>:

(2.2.5) $\quad (1-x)^{-\gamma} \Phi(f(x)) \in L(0,1)$;

(2.2.6) $\quad x^{-\gamma} \Phi(|f(e^{ix})|) \in L(0,\pi)$;

(2.2.7) $\quad \sum_{1}^{\infty} n^{\gamma-2} \Phi(n a_n) < \infty$;

(2.2.8) $\quad \sum_{1}^{\infty} n^{\gamma-2} \Phi(A_n) < \infty$,

where $A_n = a_0 + a_1 + \cdots + a_n$.

2.3 We require the following lemmas for the proof of Theorem 1.

<u>Lemma 1.</u> (Woyczynski [1]). <u>Let</u> $X = R^+$ <u>and</u> $d\mu = x^s\, dx$ $(s \leq 0)$. <u>If</u> $\Phi \in M$, <u>then</u>

$$\int_X \Phi\left(\frac{F(x)}{x}\right) d\mu \leq B \int_X \Phi(f(x))\, d\mu ,$$

where $F(x) = \int_0^x f(t)\,dt$ and $f(t) \geq 0$.

Lemma 2. (Woyczyński [1]). Let $\Phi \in \Delta_\alpha \cap YF$, $X = R^+$ and $d\mu = x^s\,dx$ ($s < -1$). If $f(x)$ is a non-negative function and $x\,f(x) \in L_\Phi(X, \mu)$, then $F(x) \in L_\Phi$, where $F(x) = \int_0^x f(t)\,dt$.

Lemma 3. (Askey and Wainger [1]). Let $\{a_n\}$ be positive and tend to zero and $\{n^{-k} a_n\}$ be monotonically decreasing for some non-negative k^*, then

$$\sum_{j=n}^{\infty} |\Delta a_j| \leq B \sum_{j=n}^{\infty} a_j/j + a_n ,$$

where B is some positive constant.

2.4 Proof of Theorem 1. We shall prove the following implications :

(i) (2.2.5) $\Longleftrightarrow$ (2.2.8) $\Rightarrow$ (2.2.6)

(ii) (2.2.6) $\Rightarrow$ (2.2.7) $\Rightarrow$ (2.2.8).

Proof of (2.2.5) $\Rightarrow$ (2.2.8). We write $(1-x)=y$, then by virtue of the fact that $(1-\frac{1}{n})^n$ $(n=1,2,\ldots)$ is an increasing sequence, we have for $\frac{1}{n+1} \leq y \leq \frac{1}{n}$, $n \geq 2$,

---

* In Lemma 3 the authors have assumed that k is a positive integer but it can be easily verified that the Lemma remains true even for $k \geq 0$.

$$f(1-y) \geq \sum_{k=0}^{n} a_k (1-y)^k \geq (1-y)^n \sum_{k=0}^{n} a_k$$

$$\geq (1-\frac{1}{n})^n \sum_{k=0}^{n} a_k \geq \frac{1}{4} A_n .$$

Thus we get

$$f(1-y) \geq \frac{1}{4} A_n \text{ for } \frac{1}{n+1} \leq y \leq \frac{1}{n} , \quad (n= 2,3,\ldots).$$

Now,

$$\sum_{1}^{\infty} n^{\gamma-2} \Phi(A_n) \leq B \sum_{1}^{\infty} \int_{n}^{n+1} t^{\gamma-2} \Phi(A_{[t]}) dt$$

$$= B \sum_{1}^{\infty} \int_{1/n+1}^{1/n} u^{-\gamma} \Phi(A_{[1/u]}) du$$

$$= B \int_{1/2}^{1} u^{-\gamma} (A_{[1/u]}) du + B \sum_{n=2}^{\infty} \int_{1/n+1}^{1/n} u^{-\gamma} \Phi(A_n) du$$

$$\leq B+B \int_{0}^{1/2} u^{-\gamma} \Phi(4 f(1-u)) du$$

$$\leq B+B \int_{0}^{1/2} u^{-\gamma} \Phi(f(1-u)) du$$

$$\leq B+B \int_{0}^{1} u^{-\gamma} \Phi(f(1-u)) du$$

$$\leq B+B \int_{0}^{1} (1-x)^{-\gamma} \Phi(f(x)) dx$$

$$< \infty .$$

<u>Proof of</u> (2.2.8) => (2.2.5).

$$\int_0^1 (1-x)^{-\gamma}\, \Phi(f(x))\, dx = \sum_{n=2}^{\infty} \int_{1-\frac{1}{n-1}}^{1-\frac{1}{n}} (1-x)^{-\gamma}\, \Phi(f(x))\, dx$$

$$= \sum_{n=2}^{\infty} \int_{1/n}^{\frac{1}{n-1}} x^{-\gamma}\, \Phi(f(1-x))\, dx$$

$$= \sum_{n=2}^{\infty} \int_{1/n}^{\frac{1}{n-1}} x^{-\gamma}\, \Phi\Big(\sum_{k=0}^{\infty} a_k (1-x)^k\Big)\, dx$$

$$\leq \sum_{n=2}^{\infty} \int_{1/n}^{\frac{1}{n-1}} x^{-\gamma}\, \Phi\Big(\sum_{k=0}^{\infty} a_k \big(1-\tfrac{1}{n}\big)^k\Big)\, dx$$

$$\leq B \sum_{n=2}^{\infty} n^{\gamma-2}\, \Phi\Big(\sum_{k=0}^{\infty} \sum_{j=nk}^{n(k+1)} a_j \big(1-\tfrac{1}{n}\big)^j\Big)$$

$$\leq B \sum_{n=2}^{\infty} n^{\gamma-2}\, \Phi\Big(\sum_{k=0}^{\infty} \big(1-\tfrac{1}{n}\big)^{nk} \sum_{j=0}^{n(k+1)} a_j\Big)$$

$$\leq B \sum_{n=2}^{\infty} n^{\gamma-2}\, \Phi\Big(\sum_{k=0}^{\infty} e^{-k} \Big(\sum_{j=0}^{n} a_j + \sum_{j=n+1}^{n(k+1)} a_j\Big)\Big)$$

$$\leq B \sum_{n=2}^{\infty} n^{\gamma-2}\, \Phi\Big(\sum_{k=0}^{\infty} e^{-k} \Big(A_n + B_2 \sum_{j=n+1}^{n(k+1)} j^{-\beta}\Big)\Big)$$

$$\leq B \sum_{n=2}^{\infty} n^{\gamma-2}\, \Phi\Big(\sum_{k=0}^{\infty} e^{-k} \big(A_n + B\, n^{1-\beta} k\big)\Big)$$

$$\leq B \sum_{n=2}^{\infty} n^{\gamma-2}\, \Phi\Big(\sum_{k=0}^{\infty} B.\ (k+1)\, e^{-k}\, A_n\Big)$$

$$\leq B \sum_{n=2}^{\infty} n^{\gamma-2}\, \Phi(A_n) < \infty .$$

Proof of (2.2.8) => (2.2.6). We shall prove that

$x^{-\gamma}\,\Phi(\,|\mathrm{Re}\; f(e^{ix})|\,)$ and $x^{-\gamma}\,\Phi(|\mathrm{Im}\; f(e^{ix})|\,)$ are both in $L\,(0,\pi)$.

Writing $D_n\,(x) = \dfrac{\sin(n+\frac{1}{2})x}{2\sin {}^{x}/_{2}}$, we have

$$\begin{aligned}
|\mathrm{Re}\; f(e^{ix})| &= \Big|\, a_0 + \sum_{k=1}^{\infty} a_k \cos kx \,\Big| \\
&\le \sum_{k=0}^{n} a_k + \Big|\sum_{k=n+1}^{\infty} a_k \cos kx\,\Big| \\
&= A_n + \Big|\sum_{k=n+1}^{\infty} \Delta a_k\, D_k(x) - a_n\, D_n(x)\,\Big| \\
&\le A_n + B\,\frac{1}{x}\sum_{k=n}^{\infty}|\Delta a_k| + B\,\frac{1}{x}\,a_n, \qquad \pi/n+1 \le x \le {}^{\pi}/n \\
&\le A_n + B\,n\Big(\sum_{k=n}^{\infty}|\Delta a_k| + a_n\Big) \\
&\le A_n + B\,n\Big(\sum_{k=n}^{\infty}\frac{a_k}{k} + a_n + a_n\Big) \quad \text{(by Lemma 3)} \\
&\le A_n + B_2\Big(n\sum_{k=n}^{\infty}\frac{1}{k^{1+\beta}} + 2\,n^{1-\beta}\Big) \\
&\le A_n + B\,n^{1-\beta} \le B\,A_n\,.
\end{aligned}$$

Now

$$\int_0^{\pi} x^{-\gamma}\,\Phi(|\mathrm{Re}\; f(e^{ix})|\,)dx = \sum_{1}^{\infty}\int_{\pi/n+1}^{\pi/n} x^{-\gamma}\,\Phi(|\,\mathrm{Re}\; f(e^{ix})|\,)dx$$

$$\leq B \sum_{1}^{\infty} n^{\gamma-2} \Phi(B A_n)$$

$$\leq B \sum_{1}^{\infty} n^{\gamma-2} \Phi(A_n)$$

$$< \infty .$$

Almost the same proof remains valid for $x^{-\gamma}$ $(|\mathrm{Im}\, f(e^{ix})|)$ $\in L(0,\pi)$ and so $x^{-\gamma}\, \Phi(|f(e^{ix})|) \in L\,(0,\pi)$, whenever $\sum_{1}^{\infty} n^{\gamma-2}\, \Phi(A_n) < \infty$.

<u>Proof of</u> (2.2.6) => (2.2.7) Let us write

$$r(t) = \mathrm{Re}\, f(e^{it}) \ , \quad R(t) = \int_0^t r(x)\, dx,$$

$$R_1(t) = \int_0^t R(x)\, dx.$$

Then

$$R_1(t) = a_0 \frac{t^2}{2} + \sum_{j=1}^{\infty} a_j .\ j^{-2}\, (1 - \cos j.t)$$

$$\geq \sum_{j=1}^{n} a_j\, j^{-2}\, (1 - \cos jt)$$

$$= 2 \sum_{j=1}^{n} j^{-2}\, a_j \sin^2 \frac{jt}{2}$$

$$\geq 2 \sum_{j=1}^{n} j^{-2}\, a_j .\ \frac{4}{\pi^2}\ \frac{j^2 t^2}{4}, \quad \pi/n+1 \leq t \leq \pi/n$$

$$\geq B\, t^2 \sum_{j=1}^{n} a_j$$

$$\geq B\, t^2\, n\, a_n .$$

Thus

$$\sum_{n=1}^{\infty} n^{\gamma-2} \Phi(n\, a_n) \leq \sum_{n=1}^{\infty} \int_{\pi/_{n+1}}^{\pi/n} x^{-\gamma} \Phi\left(\frac{1}{B} \frac{R_1(x)}{x^2}\right) dx$$

$$\leq B \sum_{n=1}^{\infty} \int_{\pi/_{n+1}}^{\pi/n} x^{-\gamma} \Phi\left(\frac{R_1(x)}{x^2}\right) dx$$

$$= B \int_0^{\pi} x^{-\gamma} \Phi\left(\frac{1}{x^2} \int_0^x R(t)\, dt\right) dx$$

$$\leq B \int_0^{\pi} x^{-\gamma} \Phi\left(\frac{1}{x^2} \int_0^x |R(t)|\, dt\right) dx$$

$$\leq B \int_0^{\pi} x^{-\gamma} \Phi\left(\frac{1}{x} \int_0^x \frac{|R(t)|}{t}\, dt\right) dx$$

$$\leq B \int_0^{\pi} x^{-\gamma} \Phi\left(\frac{|R(x)|}{x}\right) dx$$

$$\leq B \int_0^{\pi} x^{-\gamma} \Phi(|r(x)|)\, dx$$

$$\leq B \int_0^{\pi} x^{-\gamma} \Phi(|f(e^{ix})|)\, dx$$

$$< \infty,$$

by Lemma 1 and by virtue of the hypothesis.

Proof of (2.2.7) => (2.2.8). Let $a(x)$ be the function equal to a_n if $n-1 \leq x < n$, $n=1,2,\ldots$, and let $A(x)=\int_0^x a(t)\,dt$. The assumption $\Sigma\, n^{\gamma-2}\, \Phi(n\, a_n)$ implies that $t^{\gamma-2}\, \Phi(t\, a(t))$

is integrable on the positive half line, and by virtue of Lemma 2 ($s = \gamma - 2 < -1$) $t^{\gamma-2}\,\Phi(A(t))$ is integrable as well. But this is equivalent to the convergence of the series $\Sigma\, n^{\gamma-2}\,\Phi(A_n)$. Hence (2.2.7) => (2.2.8).

This proves Theorem 1.

<u>Remarks</u> : Our proof shows that the implication : (2.2.5) => (2.2.8) holds true under a less restrictive condition, namely $a_n \geq 0$. It is also known (Askey and Karlin [1]) that (2.2.8) => (2.2.5) under certain lighter conditions, namely $0 \leq \gamma < 2$, Φ increasing, positive and convex.

SECTION B

2.3 Heywood [2] proved the following result :

<u>Theorem</u> B. Suppose that $F(x) = \sum_{0}^{\infty} c_n x^n$ for $0 \leq x < 1$, $\gamma < 1$ and that there is a positive number ε such that $c_n > \frac{-K}{n^{\gamma+\varepsilon}}$, for all sufficiently large values of n, K being some positive constant. Then $(1-x)^{-\gamma} F(x) \in L\,(0,1)$ iff $\Sigma\, n^{\gamma-1}\,|c_n|$ converges.

The object of this section is to obtain a generalisation of Theorem B.

2.6 We prove the following result :

Theorem 2. *Let* $F(x) = \sum_{0}^{\infty} c_n x^n$, $0 \le x < 1$ *and* $\gamma < 1$. *Suppose that there is a positive number* ε *such that*

$$(2.6.1) \qquad c_n > \frac{-K}{n^{\gamma/p+(1-1/p)+\varepsilon}} \qquad (0 < p \le \infty)$$

for all sufficiently large values of n, *where* K *is some positive constant. Then* $(1-x)^{-\gamma} (|F(x)|)^p \in L(0,1)$ *iff* $\sum n^{\gamma-2} \left(\sum_{k=1}^{n} |c_k|\right)^p$ *converges.*

It may be remarked that in view of Lemma 6 and Abel's transformation, our Theorem includes as a special case for p≥1 the above theorem of Heywood.

2.7 We require the following lemmas to prove Theorem 2.

Lemma 4. (Titchmarsh [1] , p.58). *If* b *is a constant, then*

$$\frac{\Gamma(x)}{\Gamma(x+b)} \sim x^{-b}, \text{ as } x \to \infty.$$

Lemma 5. (Khan [2]). *Let* $F(x) = \sum_{0}^{\infty} c_n x^n$, $c_n \ge 0$, $0 \le x < 1$, $s_n = \sum_{k=1}^{n} c_k$ *and* $\gamma < 1$. *Then, for* $0 < p \le \infty$

$$\left(\int_0^1 (1-x)^{-\gamma} (F(x))^p \, dx\right)^{1/p} < \infty \text{ iff } \left(\sum_{1}^{\infty} n^{\gamma-2} s_n^p\right)^{1/p} < \infty .$$

Lemma 6. (Hardy [1], p.255). If $c > 1$, $a_n = \sum_{k=1}^{n} a_k$, $a_k \geq 0$, then $\sum_{n=1}^{\infty} n^{-c} a_n^p \leq K \sum_{n=1}^{\infty} n^{-c} (n\, a_n)^p$ $\quad (p \geq 1)$.

Lemma 7. (Konyushkov [1], p.83). If $c > 1$, $0 < p < 1$, $a_n \geq 0$ and $\{ n^{-j} a_n \}$ is monotonic decreasing for some $j > 0$, then

$$\sum_{n=1}^{\infty} n^{-c} \Big(\sum_{1}^{n} a_k \Big)^p \leq K \sum_{n=1}^{\infty} n^{-c} (n\, a_n)^p .$$

2.3 Proof of Theorem 2. We may suppose without loss of any generality that $\frac{\gamma-1}{p} + \varepsilon$ is not an integer. This will ensure the existence of the Gamma function at all relevant points. Let

$$G(x) = 2K\, \Gamma\Big(\frac{1-\gamma}{p} - \varepsilon\Big)\, (1-x)^{\frac{\gamma-1}{p} + \varepsilon}$$

$$= \sum_{n=0}^{\infty} a_n x^n , \qquad \text{for } 0 \leq x < 1.$$

Then, since

$$(1-x)^{\frac{\gamma-1}{p} + \varepsilon} = \sum_{n=0}^{\infty} \frac{\Gamma\big(n + \frac{1-\gamma}{p} - \varepsilon\big)}{\Gamma(n+1)\, \Gamma\big(\frac{1-\gamma}{p} - \varepsilon\big)}\, x^n ,$$

We have

$$(2.6.1) \qquad a_n = \epsilon K \frac{\Gamma(n+\frac{1-\gamma}{p}-\epsilon)}{\Gamma(n+1)} \sim \frac{\epsilon K}{n^{\frac{\gamma-1}{p}+1+\epsilon}}, \text{ as } n \to \infty,$$

by Lemma 4. It follows from (2.6.1) that $c_n + a_n$ is positive for all sufficiently large values of n. Since

$$F(x) + G(x) = \sum_{n=0}^{\infty} (c_n+a_n)\, x^n$$

for $0 \leq x < 1$, Lemma 5 now shows that

$$(1-x)^{-\gamma}(\, F(x)+G(x)\,)^p \in L(0,1), \text{ iff } \Sigma\, n^{\gamma-2}\Big(\sum_{k=1}^{n}(c_k+a_k)\Big)^p$$

Converges.

But, $(1-x)^{-\gamma}\, G^p(x)$ is a multiple of $(1-x)^{\epsilon p-1}$ and therefore integrable L in (0,1). Moreover (2.6.1) shows that $\Sigma\, n^{\gamma-2}\, \Big(\sum_{k=1}^{n}|a_k|\Big)^p$ is convergent by Lemmas 6 and 7 . Therefore, it follows that

$$(1-x)^{-\gamma}\,(|F(x)|)^p \in L(0,1)$$

iff

$$\Sigma\, n^{\gamma-2}\,\Big(\sum_{k=1}^{n}|c_k|\Big)^p < \infty .$$

Thus the theorem is proved.

Chapter III

ON FOURIER COEFFICIENTS WITH POSITIVE FUNCTIONS

3.1 A function $\phi(x)$ is said to belong to the class $L(p,\alpha)$ (Askey and Wainger [1]) if

$$\int_0^\pi |\phi(x)|^p (\sin x)^{\alpha p} dx < \infty ,$$

α is a real number and $p > 0$.

We define the norm of a function $\phi(x) \in L(p,\alpha)$ as :

$$\|\phi(x)\|_{p,\alpha} = \left\{ \int_0^\pi |\phi(x)|^p (\sin x)^{\alpha p} dx \right\}^{1/p} .$$

It is evident that $L(p,\alpha) \Rightarrow L^p$ for $\alpha < 0$, $L^p \Rightarrow L(p,\alpha)$ for $\alpha > 0$ and $L(p,\alpha) \equiv L^p$ where $\alpha = 0$.

Throughout this Chapter, K with or without suffixes, denotes a positive constant, not necessarily the same at each occurrence.

3.2 Concerning the Fourier series of positive functions Askey and Boas [1] proved the following theorems.

<u>Theorem</u> A. Let $G(x)\downarrow$ on $(0,\pi)$, G bounded below and

$$\int_0^{\pi} x\, dG(x) ,$$

finite, so that dG has generalized sine coefficients

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin nx\, dG(x)$$

If $1 < p < \infty$ and $\frac{1}{p} < \gamma < 1 + \frac{1}{p}$, then $\{n^{-\gamma} b_n\} \in \ell^p$ if, and only if

$$t^{\gamma-1-2/p} \int_0^{t} x\, dG(x) \in L^p .$$

<u>Theorem</u> B. Let $F(x)\downarrow$ on $(0,\pi)$, F bounded below and

$$\int_0^{\pi} x^2\, dF(x)$$

finite. Let

$$a_n = -\frac{2}{\pi}\int_0^{\pi}(1-\cos nx)\, dF(x)$$

be the generalized cosine coefficients of dF. If $1 < p < \infty$ and $\frac{1}{p} < \gamma < 2+ {}^1/p$, then $\{n^{-\gamma} a_n\} \in \ell^p$ if, and only if

$$t^{\gamma-2-2/p} \left(\int_0^{t} u^2\, dF(u)\right) \in L^p .$$

<u>Theorem</u> C. If $-\frac{1}{p'} < \gamma < \frac{1}{p}$ and $\{n^{-\gamma} a_n\} \in \ell^p$, where a_n are the Fourier coefficients of dF with F monotonic, then

$$t^{\gamma - 2/p} \left[F(t) - F(0)\right] \in L^p, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

<u>Theorem</u> D. If $-\frac{1}{p'} < \gamma < \frac{1}{p'}$, a_n are the Fourier cosine coefficients of f and $t^{\gamma-2/p} \left(\int_0^t x \, |df(x)| \right) \in L^p$, then $\{n^{-\gamma} a_n\} \in \ell^p$.

Recently, Mazhar and Khan [1] have generalised the above theorems in the following form.

<u>Theorem</u> E. Let $G(x)$ satisfy the conditions of Theorem A. If $1 < p < \infty$, $\lambda(x)$ is a positive function such that

(3.2.1) $x^{1+\delta} \lambda(x)$ is decreasing for some small $\delta > 0$,

(3.2.2) $x^{p+1-\delta} \lambda(x) \uparrow +\infty$ for some small $\delta > 0$, as $x \to \infty$, then

$$\left\{ \lambda^{1/p}(n) \, b_n \right\} \in \ell^p$$

if, and only if,

$$\lambda^{1/p} (\pi/t) \, t^{-1-2/p} \int_0^t x \, dG(x) \in L^p.$$

Theorem F. Let $F(x)$ satisfy the conditions of Theorem B and let

$$a_n = -\frac{2}{\pi}\int_0^{\pi}(1-\cos nx)\, dF(x)$$

be the generalized cosine coefficients of dF. If $1 < p < \infty$ and $\lambda(x)$ is a positive function such that

(3.2.3) $x^{1+\delta}\lambda(x)$ is decreasing for some small $\delta > 0$,

(3.2.4) $x^{2p+1-\delta}\lambda(x) \uparrow +\infty$ for some small $\delta > 0$, then

$$\{\lambda^{1/p}(n)\, a_n\} \in \ell^p$$

if, and only if

$$\lambda^{1/p}(\pi/t)\, t^{-2-2/p}\int_0^t x^2\, dF(x) \in L^p.$$

Theorem G. If a_n are the Fourier coefficients of dF with F monotonic and $\mu(x)$ is a positive function such that

(3.2.5) $x^{1+\delta-p}\mu(x) \downarrow 0$ for some small $\delta > 0$, $x \to \infty$,

(3.2.6) $x^{1-\delta}\mu(x) \uparrow +\infty$ for some small $\delta > 0$, $x \to \infty$, and

$$\{\mu^{1/p}(n)\, a_n\} \in \ell^p,$$

then

$$\mu^{1/p}(\pi/t)\, t^{-2/p}\, [F(t) - F(0)] \in L^p.$$

Theorem H. If a_n are the Fourier cosine coefficients of f and if

$$\mu^{1/p}(\pi/t)\, t^{-2/p} \int_0^t x\, |df(x)| \in L^p,$$

then

$$\{\mu^{1/p}(n)\, a_n\} \in \ell^p,$$

where $\mu(x)$ satisfies the same conditions as in Theorem G.

The object of this chapter is to prove the following theorems in which the class L^p has been replaced by a wider class $L(p,\alpha)$.

3·3. We prove the following Theorems.

Theorem 1. Let $G(x)$ satisfy the conditions of Theorem A. If $1 < p < \infty$, $-1 < \alpha p < p-1$, then

$$\{n^{-\alpha}\, \lambda^{1/p}(n)\, b_n\} \in \ell^p$$

if, and only if

$$\lambda^{1/p}(\pi/t)\, t^{-1-2/p} \left(\int_0^t x\, d\, G(x)\right) \in L(p,\alpha).$$

where $\lambda(x)$ *is a positive function such that*

(3.3.1) $x^{1+\delta-\alpha p}\,\lambda(x)\downarrow$ *for some small* $\delta > 0$,

(3.3.2) $x^{p+1-\delta}\,\lambda(x)\uparrow +\infty$ *for some small* $\delta > 0$, $x \to \infty$.

Theorem 2. Let $F(x)$ *satisfy the conditions of Theorem* B, *and let*

$$a_n = -\frac{2}{\pi}\int_0^{\pi}(1 - \cos nx)\; dF(x)$$

be the generalized cosine coefficients of dF. If $1 < p < \infty$, *and* $-1 < \alpha p < p-1$, *then*

$$\{ n^{-\alpha}\; \lambda^{1/p}(n)\; a_n \} \in \ell^p,$$

if, and only if

$$\lambda^{1/p}(\pi/t)\; t^{-2-2/p}\; (\int_0^t x^2\; dF(x)) \in L(p,\alpha),$$

where $\lambda(x)$ *is a positive function such that*

(3.3.3) $x^{1+\delta-\alpha p}\,\lambda(x)\downarrow$ *for some small* $\delta > 0$,

(3.3.4) $x^{2p+1-\delta}\,\lambda(x)\uparrow +\infty$ *for some small* $\delta > 0$.

Theorem 3. If a_n are the Fourier coefficients of dF with F monotonic and $\mu(x)$ is a positive function such that

(3.3.5) $x^{1+\delta-p-\alpha p}\mu(x)\downarrow 0$ for some small $\delta > 0$, $x\to\infty$, and

(3.3.6) $x^{1-\eta}\mu(x)\uparrow +\infty$ for some small η, $x\to\infty$.

If $\{n^{-\alpha}\mu^{1/p}(n)\, a_n\}\in \ell^p$, then

$$\mu^{1/p}(\pi/t)\; t^{-2/p}\,[F(t)-F(0)] \in L(p,\alpha),$$

where $1 < p < \infty$ and α is a real number such that $-1<\alpha p<\eta$.

Theorem 4. If a_n are the Fourier cosine coefficients of f and if

$$\mu^{1/p}(\pi/t)\,t^{-2/p}\Big(\int_0^t x\,|df(x)|\Big) \in L(p,\alpha),\ 1<p<\infty,\ -1<\alpha p< p-1,$$

then $\{n^{-\alpha}\mu^{1/p}(n)\, a_n\}\in \ell^p$, where $\mu(x)$ satisfies the same conditions as in Theorem 3.

3.4. The following lemmas are pertinent for the proof of our theorems.

Lemma 1. (Askey and Boas [1]). If $G(x)\downarrow$, $\int_0^\pi x|dG(x)| <\infty$, and b_n are the generalised sine coefficients of dG, then

$$\Big| \sum_{\nu=1}^{n}{}' b_\nu \Big| \geq K n^2 \int_0^{\pi/n} x \, |dG(x)| \ ,$$

where Σ' has the last term of the sum halved.

Lemma 2. (Khan [3]). Let $\lambda(n)$ be a positive monotonic decreasing sequence such that

$$\sum_{k=n}^{\infty} \lambda(k) = O(n\,\lambda(n)) \ , \ n \to \infty .$$

Let $s_n = \sum_{k=1}^{n} a_k$, $a_k \geq 0$ and $\sum_{n=1}^{\infty} \lambda(n) \ (n\, a_n)^p < \infty$, $p>1$, then $\sum_{n=1}^{\infty} \lambda(n) \ s_n^p < \infty$ and

$$\sum_{n=1}^{\infty} \lambda(n) \ s_n^p \leq K \sum_{n=1}^{\infty} \lambda(n) \ (n\, a_n)^p .$$

Lemma 3. (Khan [3]). Suppose that $\phi(x)$ increases and is bounded with $\phi(+0) = 0$. Let $\psi(x)$ be a function such that

$$\int_0^{\pi} \psi(u) \ \phi^p(u) \ du < \infty ,$$

then

$$\int_0^{\pi} \psi(u) \ u^{sp} \ (\int_u^{\pi} x^{-s} \ d\phi(x))^p \ du < \infty ,$$

where $s > 0$, $p > 1$, and

(3.4.1) $x^{1-\delta+\alpha p}\,\psi(x)$ <u>is a positive increasing function for some small</u> $\delta > 0$.

3.5 <u>Proof of Theorem</u> 1. <u>Necessity</u>. We have by Lemma 1 and 2

$$\int_0^{\pi} u^{-p-2}\,\lambda(\pi/u)\,\left(\int_0^u x\,|dG(x)|\right)^p \sin^{\alpha p} u\,du$$

$$= \sum_{n=2}^{\infty} \int_{\pi/n+1}^{\pi/n} u^{-p-2}\,\lambda(\pi/u)\,\sin^{\alpha p} u\,\left(\int_0^u x\,|dG(x)|\right)^p du + \int_{\pi/2}^{\pi} u^{-p-2}\,\lambda(\pi/u)\sin^{\alpha p} u\left(\int_0^u x\,|dG(x)|\right)^p du$$

$$\leq K \sum_{n=2}^{\infty} \int_{\pi/n+1}^{\pi/n} u^{\alpha p-p-2}\,\lambda(\pi/u)\left(\int_0^u x\,|dG(x)|\right)^p du + \left(\frac{\pi}{2}\right)^{-p-2}\lambda(2)\left(\int_0^{\pi} x\,|dG(x)|\right)^p \int_{\pi/2}^{\pi} \sin^{\alpha p} u\,du$$

$$\leq K \sum_{n=2}^{\infty} n^{p-\alpha p}\lambda(n)\left(\int_0^{\pi/n} x\,|dG(x)|\right)^p + K\int_0^{\pi/2} \sin^{\alpha p} u\,du$$

$$\leq K \sum_{n=2}^{\infty} n^{-p-\alpha p}\,\lambda(n)\,\left(\sum_{k=1}^{n}{}' \,|b_k|\right)^p + K\int_0^{\pi/2} u^{\alpha p}\,du$$

$$\leq K \sum_{n=2}^{\infty} n^{-p-\alpha p}\,\lambda(n)\,\left(n\,|b_n|\right)^p + K$$

$$= K \sum_{n=2}^{\infty} n^{-\alpha p} \lambda(n) \, |b_n|^p + K$$

$$< \infty .$$

Sufficiency. Let $\| a_n \|$ denote the norm of a_n in l^p space, that is to say

$$\| a_n \| = \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{1/p}.$$

We have

$$\| \lambda^{1/p}(n) \, n^{-\alpha} \, b_n \| = K \| \lambda^{1/p}(n) \, n^{-\alpha} \int_0^{\pi} \sin nx \, dG(x) \|$$

$$= K \| \lambda^{1/p}(n) . n^{-\alpha} \left(\int_0^{1/n} + \int_{1/n}^{\pi} \right) \sin nx \, dG(x) \|$$

$$\leq K \| \lambda^{1/p}(n) n^{-\alpha} \int_0^{1/n} \sin nx \, dG(x) \| +$$

$$+ K \| n^{-\alpha} \lambda^{1/p}(n) \int_{1/n}^{\pi} \sin nx \, dG(x) \|$$

$$\leq K \| \lambda^{1/p}(n) n^{1-\alpha} \int_0^{1/n} x \, |dG(x)| \| +$$

$$+ K \| \lambda^{1/p}(n) n^{-\alpha} \int_{1/n}^{\pi} |dG(x)| \|$$

$$\leq K \left\{ \sum_{n=1}^{\infty} \lambda(n) n^{p-\alpha p} \left(\int_0^{\pi/n} x \, |dG(x)| \right)^p \right\}^{1/p} +$$

$$+ K\left\{ \sum_{n=1}^{\infty} \lambda(n) n^{-\alpha p} \left(\int_{1/n}^{\pi} |dG(x)| \right)^{p} \right\}^{1/p}$$

$$\leq K \left(\sum_{n=3}^{\infty} \int_{\pi/n}^{\pi/n-1} \lambda(\pi/t).\ t^{-p-2+\alpha p} \left(\int_{0}^{t} x\, |dG(x)| \right)^{p} dt \right)^{1/p} +$$

$$+ K \left(\sum_{n=2}^{\infty} n^{-\alpha p} \lambda(n) \left(\int_{1/n}^{\pi/n} + \int_{\pi/n}^{\pi} \right)^{p} \right)^{1/p} + K$$

$$\leq K \left(\int_{0}^{\pi/2} t^{-p-2+\alpha p} \lambda(\pi/t) \left(\int_{0}^{t} x\, | dG(x)| \right)^{p} dt \right)^{1/p} +$$

$$+ K \left(\sum_{n=2}^{\infty} n^{-\alpha p} \lambda(n) \left(\int_{1/n}^{\pi/n} | dG(x)| \right)^{p} \right)^{1/p} +$$

$$+ K \left(\sum_{n=2}^{\infty} n^{-\alpha p} \lambda(n) \left(\int_{\pi/n}^{\pi} | dG(x)| \right)^{p} \right)^{1/p} + K$$

$$= I_1^{1/p} + I_2^{1/p} + I_3^{1/p} + K \text{ , say .}$$

Since

$$\int_{0}^{\pi} \lambda(\pi/t)\ t^{-2-p} \sin^{\alpha p} t \left(\int_{0}^{t} x\ | dG(x)| \right)^{p} dt < \infty ,$$

we have

$$I_1 = K \int_{0}^{\pi/2} \lambda(\pi/t)\ t^{-p-2+\alpha p} \left(\int_{0}^{t} x\ |dG(x)| \right)^{p} dt$$

$$\leq K \int_0^{\pi/2} \lambda(\pi/t)\, t^{-p-2} \sin^{\alpha p} t \left(\int_0^t x\, |dG(x)|\right)^p dt < \infty.$$

Also we have

$$I_2 = \sum_{n=2}^{\infty} \lambda(n)\, n^{-\alpha p} \left(\int_{1/n}^{\pi/n} |dG(x)|\right)^p$$

$$\leq K + K \sum_{n=3}^{\infty} \lambda(n)\, n^{p-\alpha p} \left(\int_{1/n}^{\pi/n} x\, |dG(x)|\right)^p$$

$$\leq K + K \sum_{n=3}^{\infty} \int_{\pi/n}^{\pi/n-1} \lambda(\pi/t)\, t^{-p-2+\alpha p} \left(\int_{1/n}^{t} x\, |dG(x)|\right)^p dt$$

$$\leq K + K \int_0^{\pi/2} \lambda(\pi/t)\, t^{-p-2+\alpha p} \left(\int_0^t x\, |dG(x)|\right)^p dt$$

$$< \infty .$$

Now

$$I_3 \leq K \sum_{n=1}^{\infty} n^{-\alpha p} \lambda(n) \left(\int_{\pi/n}^{\pi} |dG(x)|\right)^p$$

$$\leq K \sum_{n=1}^{\infty} \int_{\pi/n+1}^{\pi/n} t^{\alpha p-2} \lambda(\pi/t) \left(\int_t^{\pi} |dG(x)|\right)^p dt$$

$$= K \int_0^{\pi} t^{\alpha p-2} \lambda(\pi/t) \left(\int_t^{\pi} |dG(x)|\right)^p dt.$$

By taking $\phi(t) = \int_0^t x\,|dG(x)|$, $s=1$, $\psi(t) = t^{-p-2+\alpha p}\,\lambda(\pi/t)$ and applying Lemma 3, we find, in view of I_1 , that

$$\int_0^\pi \lambda(\pi/t).t^{-p-2+\alpha p}.t^p \left(\int_t^\pi x^{-1}\, x\,|dG(x)|\,\right)^p dt < \infty ,$$

that is to say

$$I_3 < \infty .$$

Hence

$$\| \lambda^{1/p}(n)\, n^{-\alpha}\, b_n \| < \infty .$$

This proves Theorem 1.

3.6 <u>Proof of Theorem</u> 2. The proof of this theorem is similar to that of Theorem 1. However, for the sake of completeness we include a proof here.

<u>Necessity.</u> We have in view of (3.3.3) and (3.3.4)

$$\int_0^\pi \lambda(\pi/t)\; t^{-2p-2}\; \sin^{\alpha p} t\; \left(\int_0^t u^2\, |dF(u)|\,\right)^p dt$$

$$= \sum_{n=2}^{\infty} \int_{\pi/_{n+1}}^{\pi/n} \lambda(\pi/t)\, t^{-2p-2}\, \sin^{\alpha p} t \left(\int_0^t u^2\,|dF(u)|\,\right)^p dt +$$

$$+ \int_{\pi/2}^{\pi} \lambda(\pi/t)\, t^{-2p-2} \sin^{\alpha p} t \left(\int_0^t u^2 \, |dF(u)| \right)^p dt$$

$$\leq K \sum_{n=2}^{\infty} \int_{\pi/_{n+1}}^{\pi/n} \lambda(\pi/t)\, t^{\alpha p-2p-2} \left(\int_0^t u^2 \, |dF(u)|\right)^p dt +$$

$$+ \lambda(2)(\pi/2)^{-2p-2} \left(\int_0^{\pi} u^2 \, |dF(u)| \right)^p \int_{\pi/2}^{\pi} \sin^{\alpha p} t \, dt$$

$$\leq K \sum_{n=2}^{\infty} \lambda(n) n^{2p-\alpha p} \left(\int_0^{\pi/n} u^2 \, |dF(u)|\right)^p + K \int_0^{\pi/2} \sin^{\alpha p} t \, dt$$

$$\leq K \sum_{n=2}^{\infty} \lambda(n)\, n^{-\alpha p} \, |a_n|^p + K \int_0^{\pi/2} t^{\alpha p} \, dt$$

$$< \infty ,$$

by virtue of the hypothesis and the fact that

$$|a_n| = \frac{2}{\pi} \int_0^{\pi} (1 - \cos nx) \, |dF(x)|$$

$$= \frac{2}{\pi} \cdot 2 \int_0^{\pi} \sin^2 \frac{nx}{2} \, |dF(x)|$$

$$\geq K \int_0^{\pi/n} \sin^2 \frac{nx}{2} \, |dF(x)|$$

$$\geq K \int_0^{\pi/n} n^2 x^2 \, |dF(x)|$$

$$= K n^2 \int_0^{\pi/n} x^2 \, |dF(x)| .$$

Sufficiency.

$$\| \lambda^{1/p}(n).n^{-\alpha}a_n\| = K \|\lambda^{1/p}(n)n^{-\alpha}\int_0^{\pi}(1-\cos nx)dF(x)\|$$

$$\leq K\|\lambda^{1/p}(n)n^{2-\alpha}\int_0^{1/n}x^2\ |dF(x)|\ \| +$$

$$+ K\|\lambda^{1/p}(n)n^{-\alpha}\int_{1/n}^{\pi}|dF(x)|\ \|$$

$$\leq K\Big(\sum_{n=1}^{\infty}\lambda(n)n^{2p-\alpha p}\Big(\int_0^{\pi/n}x^2\ |dF(x)|\ \Big)^p\ \Big)^{1/p} +$$

$$+ K\Big(\sum_{n=1}^{\infty}\lambda(n)n^{-\alpha p}\Big(\int_{1/n}^{\pi}|dF(x)|\Big)^p\ \Big)^{1/p}$$

$$\leq K\Big(\sum_{n=3}^{\infty}\int_{\pi/n}^{\pi/n-1}\lambda(\pi/t)t^{-2p-2+\alpha p}\Big(\int_0^t x^2\ |dF(x)|\ \Big)^p dt\Big)^{\frac{1}{p}}$$

$$+ K\Big(\sum_{n=2}^{\infty}\lambda(n)n^{-\alpha p}\Big(\int_{1/n}^{\pi/n}|dF(x)|\ \Big)^p\ \Big)^{1/p}$$

$$+ K\Big(\sum_{n=2}^{\infty}\lambda(n)n^{-\alpha p}\Big(\int_{\pi/n}^{\pi}|dF(x)|\ \Big)^p\ \Big)^{1/p} + K.$$

$$\leq J_1^{1/p} + J_2^{1/p} + J_3^{1/p} + K\ , \text{ say.}$$

Since by virtue of the hypothesis

$$\int_0^{\pi} \lambda(\pi/t)\, t^{-2p-2} \sin^{\alpha p} t \left(\int_0^t x^2 |dF(x)|\right)^p dt < \infty ,$$

We have

$$J_1 = K \int_0^{\pi/2} \lambda(\pi/t)\, t^{-2p-2+\alpha p} \left(\int_0^t x^2 \,|\, dF(x) \,|\right)^p dt$$

$$\leq K \int_0^{\pi/2} \lambda(\pi/t)\, t^{-2p-2} \sin^{\alpha p} t \left(\int_0^t x^2 \,|\, dF(x) \,|\right)^p dt$$

$$< \infty .$$

Also we have

$$J_2 = K \sum_{n=2}^{\infty} \lambda(n)\, n^{-\alpha p} \left(\int_{1/n}^{\pi/n} |dF(x)| \right)^p$$

$$\leq K \sum_{n=2}^{\infty} \lambda(n)\, n^{2p-\alpha p} \left(\int_{1/n}^{\pi/n} x^2 \,|dF(x)| \right)^p$$

$$\leq K \sum_{n=2}^{\infty} \int_{\pi/n}^{\pi/n-1} \lambda(\pi/t)\, t^{-2p-2+\alpha p} \left(\int_{1/n}^{t} x^2 \,|dF(x)| \right)^p dt$$

$$\leq K \int_0^{\pi} \lambda(\pi/t)\; t^{-2p-2+\alpha p} \left(\int_0^t x^2 \,|dF(x)| \right)^p dt$$

$$< \infty .$$

Now

$$J_3 \leq K \sum_{n=1}^{\infty} n^{-\alpha p} \lambda(n) \left(\int_{\pi/n}^{\pi} |dF(x)| \right)^p$$

$$\leq K \sum_{n=1}^{\infty} \int_{\pi/n+1}^{\pi/n} t^{\alpha p-2} \lambda(\pi/t) \left(\int_t^{\pi} |dF(x)| \right)^p dt$$

$$\leq K \int_0^{\pi} t^{\alpha p-2} \lambda(\pi/t) \left(\int_t^{\pi} |dF(x)| \right)^p dt.$$

By taking $\phi(t) = \int_0^t x^2 \, |dF(x)|$, $s = 2$, $\psi(t) = \lambda(\pi/t) t^{-2p-2+\alpha p}$,

and applying Lemma 3, we observe that

$$\int_0^{\pi} \lambda(\pi/t) t^{-2p-2+\alpha p} . t^{2p} \left(\int_t^{\pi} x^{-2} x^2 \, |dF(x)| \right)^p dt < \infty ,$$

that is to say, $J_3 < \infty$.

Hence, $\left\{ n^{-\alpha} \lambda^{1/p}(n) \, a_n \right\} < \infty$.

This completes the proof of Theorem 2.

3.7 Proof of Theorem 3. Since $\left\{ \mu^{1/p}(n) n^{-\alpha} a_n \right\} \in \ell^p$,

we have

$$\sum_{n=1}^{\infty} n^{-1} |a_n| = \sum_{n=1}^{\infty} n^{-1+\alpha} \mu^{-1/p}(n) . n^{-\alpha} \mu^{1/p}(n) \, |a_n|$$

$$\leq \left(\sum_{n=1}^{\infty} n^{-p'(1-\alpha)} \mu^{-p'/p}(n) \right)^{1/p'} \left(\sum_{n=1}^{\infty} n^{-\alpha p} \mu(n) \, |a_n|^p \right)^{1/p}$$

$$< \infty ,$$

by the condition (3.3.6) and $\alpha p < \eta$. Hence $\Sigma\, n^{-1} a_n \sin nx$ is the Fourier series of a function F so that $\Sigma\, a_n \cos nx$ is the Fourier-Stieltjes series of $F(x)$. Suppose F is an increasing function.

Now

$$\frac{a_o}{2} + \sum_{k=1}^{n} a_k = \frac{2}{\pi} \int_0^\pi \frac{\sin(n+\frac{1}{2})x}{2\sin {x}/{2}}\, dF(x)$$

$$= \frac{1}{\pi} \int_0^\pi \sin nx \cot \frac{x}{2}\, dF(x) + \frac{1}{\pi} \int_0^\pi \cos nx\, dF(x)$$

$$= \alpha_n + \beta_n \text{, say.}$$

Using (3.3.5), Lemma 2 and the hypothesis, we have

$$\sum_{n=1}^{\infty} \mu(n) n^{-\alpha p - p} |\alpha_n + \beta_n|^p = \sum_{n=1}^{\infty} \mu(n) n^{-\alpha p - p} \left|\frac{1}{2} a_o + \sum_{k=1}^{n} a_k\right|^p$$

$$\leq K + K \sum_{n=1}^{\infty} \mu(n) n^{-\alpha p - p} \left(\sum_{k=1}^{n} |a_k|\right)^p$$

$$< K + K \sum_{n=1}^{\infty} \mu(n) n^{-\alpha p - p} (n|a_n|)^p$$

$$= K + K \sum_{n=1}^{\infty} \mu(n)\, n^{-\alpha p} |a_n|^p$$

$$< \infty .$$

Also $\beta_n = O(1)$ and hence ,

$$\sum_{n=1}^{\infty} \mu(n)\, n^{-p-\alpha p} \,|\beta_n|^p \leq K \sum_{n=1}^{\infty} \mu(n)\, n^{-p-\alpha p} < \infty .$$

From this it follows that $\sum_{n=1}^{\infty} \mu(n)\, n^{-\alpha p-p} \,|a_n|^p < \infty$.

Let

$$dG(x) = -\cot {}^{x}/_{2}\, dF(x) , \qquad 0 \leq x \leq {}^{\pi}/_{2} \quad \text{and}$$

$$dG(x) = 0, \quad {}^{\pi}/_{2} < x \leq \pi .$$

$$a_n = -\frac{1}{\pi} \int_0^{\pi} \sin nx \, dG(x) + O(1).$$

We have already proved that $\sum_{n=1}^{\infty} \mu(n)\, n^{-\alpha p-p} \,|a_n|^p < \infty$ and $\sum_{n=1}^{\infty} \mu(n)\, n^{-\alpha p-p} < \infty$.

Also

$$\int_0^{\pi} x \,|dG(x)| \leq \int_0^{\pi} x \cot {}^{x}/_{2}\, dF(x)$$

$$\leq K \int_0^{\pi} dF(x)$$

$$< \infty .$$

Thus, if we put $\lambda(n) = n^{-p}\mu(n)$ in Theorem 1, we find that all the conditions are satisfied. Hence, by Theorem 1

$$\mu^{1/p}({}^{\pi}/_{t})\ t.\ t^{-2/p-1} \left(\int_0^{t} x \, dG(x)\right) \in L\,(p,\alpha).$$

Now

$$\int_0^t dF(x) = - \int_0^t \tan {}^x/_2 \, dG(x)$$

$$\leq \int_0^t x \, |dG(x)| \; ,$$

that is to say

$$(F(t) - F(0)) \leq \int_0^t x \, |dG(x)| \; .$$

Hence $\mu^{1/p}(\pi/t) \, t^{-2/p} \, [F(t) - F(0)] \in L(p,\alpha)$.

This proves Theorem 3.

3.8 Proof of Theorem 4. We are given that

$$\mu^{1/p}(\pi/t) \, t^{-2/p} \, (\int_0^t x \, |df(x)| \,) \in L(p,\alpha).$$

Also

$n \, a_n = - \frac{2}{\pi} \int_0^\pi \sin nt \, df(t)$, hence by virtue of Theorem 1

$\{ \mu^{1/p}(n) n^{-\alpha} a_n \} \in \ell^p$, since sufficiency part of Theorem 1 is true even when $dG(x)$ is replaced by $|dG(x)|$, where $G(x)$ is not necessarily monotonic.

This proves Theorem 4 .

Chapter IV

INTEGRABILITY THEOREMS FOR TRIGONOMETRIC SERIES WITH QUASI-MONOTONE COEFFICIENTS

4.1. Let $f(x)$ and $g(x)$ be defined by the following trigonometric series :

$$(4.1.1) \qquad f(x) = \sum_{1}^{\infty} a_n \cos nx ,$$

$$(4.1.2) \qquad g(x) = \sum_{1}^{\infty} a_n \sin nx .$$

A sequence $\{ a_n \}$ of non-negative numbers is said to be quasi-monotone (Shah [1] ; Szász [1]) if for some $\alpha > 0$,

$$a_{n+1} \leq a_n \left(1 + \frac{\alpha}{n} \right)$$

for all $n > n_o(\alpha)$, where $n_o(\alpha)$ is a positive number depending upon α.

An equivalent definition of quasi-monotone sequence (Shah [2]) is that $n^{-\beta} a_n \downarrow 0$ for some $\beta > 0$.

It is said to be a quasi-monotone of $\alpha = \alpha_o$ (Yong; [2]) if

$$a_{n+1} \leq a_n \left(1 + \frac{\alpha_o}{n} \right).$$

A positive function $L(x)$ is said to be "slowly increasing" in the sense of Karamata [1] if it is continuous for $x \geq 0$ and

$$\lim_{x\to\infty} \frac{L(tx)}{L(x)} = 1 \quad \text{for every fixed } t > 0.$$

Some of the important properties of such functions are as follows. 1)

P_1 : $\dfrac{L(tx)}{L(x)} \to 1$ as $x \to \infty$, uniformly for $0 < a \leq t \leq b < \infty$.

P_2 : $x^{\alpha} L(x) \to \infty$, $x^{-\alpha} L(x) \to 0$ as $x \to \infty$ for every $\alpha > 0$.

P_3 : If we write for some $\alpha > 0$,

$$\bar{L}_1(x) = x^{-\alpha} \operatorname*{Max}_{0 \leq t \leq x} \{t^{\alpha} L(t)\}, \quad \underline{L}_1(x) = x^{\alpha} \operatorname*{Min}_{0 < t \leq x} \{t^{-\alpha} L(t)\},$$

$$\bar{L}_2(x) = x^{\alpha} \operatorname*{Max}_{x \leq t \leq \infty} \{t^{-\alpha} L(t)\}, \quad \underline{L}_2(x) = x^{-\alpha} \operatorname*{Min}_{x \leq t < \infty} \{t^{\alpha} L(t)\},$$

then $\bar{\underline{L}}_k(x) \sim L(x)$ as $x \to \infty$ for $k = 1, 2$.

P_4 : For $\alpha > 0$, we have

$$L(tu) \leq A_1 t^{-\alpha} L(u) \quad \text{for every } u \geq 0 \text{ and } 0 < t \leq 1,$$

$$L(u/t) \leq A_2 t^{-\alpha} L(u) \quad \text{for every } u \geq 0 \text{ and } 0 < t \leq 1,$$

1) P_1 , P_2, P_3 are due to J.Karamata[1] where P_4 is due to Igari [1].

where A_1 and A_2 are positive constants depending on α and L.

A function $\phi(x)$ is said to belong to the class $L(p,\alpha)$ (Askey and Wainger [1]) if

$$\int_0^{\pi} |\phi(x)|^p (\sin x)^{\alpha p}\, dx < \infty,\ p > 0,\ \alpha \text{ being any real number.}$$

We define the norm of a function $\phi(x) \in L(p,\alpha)$ as :

$$\| \phi(x) \|_{p,\alpha} = \left\{ \int_0^{\pi} |\phi(x)|^p (\sin x)^{\alpha p}\, dx \right\}^{1/p}.$$

4.2 Concerning the integrability of trigonometric series Igari [1] , in 1960, proved the following theorems.

<u>Theorem A.</u> Suppose that $a_n \downarrow 0$, $p \geq 1$ and $-1 < \lambda < 0$. Then a necessary and sufficient condition that

$$\sum_{n=1}^{\infty} n^{-1+p+p\lambda} L(n)\, a_n^p$$

should converge is that

$$x^{-1-\lambda p} L\left({}^1/x \right) f^p(x) \in L\,(0,\pi).$$

<u>Theorem B.</u> Suppose that $a_n \downarrow 0$, $p \geq 1$ and $-1 < \lambda < 1$. Then a necessary and sufficient condition that

$$\sum_{n=1}^{\infty} n^{-1+p+p\lambda} L(n)\, a_n^p$$

should converge is that

$$x^{-1-\lambda p} L(1/x)\, g^{p}(x) \in L(0, \pi).$$

Theorem A is similar to Theorem B. The only difference between the two is that in the latter case range of λ has been extended to $\lambda < 1$.

These theorems were subsequently extended by Yong [2] to quasi-monotone sequences. His theorems are as follows :

Theorem C. Let $\{a_n\}$ be quasi-monotone of $\alpha < 1$ and such that $M_2 \geq n^{\beta} L_1(n)\, a_n \geq M_1 > 0$ with some $\beta > 0$ $(n=1,2,\ldots)$. If $p \geq 1$ and $1-p < \lambda < 1$, then $x^{-\lambda} L_2(1/x)\, f^{p}(x) \in L(0,\pi)$ iff $\sum_1^{\infty} n^{\lambda+p-2} L_2(n)\, a_n^{p}$ converges.

Theorem D. Let $\{a_n\}$ be quasi-monotone such that $M_2 \geq n^{\beta} L_1(n) a_n \geq M_1 > 0$ with some $\beta > 0$ $(n=1,2,\ldots)$. If $p \geq 1$ and $1-p<\lambda<1+p$, then $x^{-\lambda} L_2(1/x)\, g^{p}(x) \in L(0,\pi)$ iff $\sum_1^{\infty} n^{\lambda+p-2} L_2(n) a_n^{p}$ converges.

Concerning $L(p,\alpha)$ class, Askey and Wainger [1] in 1966 proved the following theorems.

Theorem E. Let $f(x) \in L(p,\alpha)$ with $1 \leq p < \infty$, $-1 < \alpha p < p-1$. Let $f(x) \sim \sum_1^{\infty} a_n \cos nx$ with $a_n \geq 0$ and $A_n = \sum_{j=[n/2]}^{n} a_j$, then

$$\sum_1^{\infty} n^{-2-\alpha p} A_n^{p} < \infty,$$

and

$$\sum_{1}^{\infty} n^{-2-\alpha p} A_n^p \leq B(\alpha,p) \, \|f\|_{p,\alpha}^p .$$

__Theorem F__. Let $1 \leq p < \infty$, $-1 < \alpha p < p-1$. Suppose that $\{a_n\}$ is a sequence of numbers such that $a_n \to 0$ and

$$\left\{ \sum_{n=1}^{\infty} n^{p-\alpha p-2} \left(\sum_{k=n}^{\infty} |\Delta a_k| \right)^p \right\}^{1/p} < \infty ,$$

then $f(x) \sim \sum_{1}^{\infty} a_n \cos nx$ is in $L(p,\alpha)$ class and

$$\|f\|_{p,\alpha}^p \leq B(\alpha,p) \sum_{n=1}^{\infty} n^{p-\alpha p-2} \left(\sum_{k=n}^{\infty} |\Delta a_k| \right)^p ,$$

where $\Delta a_k = a_k - a_{k+1}$.

From Theorems E and F they deduced the following interesting result.

__Theorem G__. Let $\{a_n\}$ be a positive sequence tending to zero and $\{n^{-k} a_n\}$ be monotonically decreasing for some non-negative integer k. Let $1 \leq p < \infty$ and $-1 < \alpha p < p-1$, then a necessary and sufficient condition that $f(x) \in L(p,\alpha)$ is that

$$\sum_{1}^{\infty} n^{p-\alpha p-2} a_n^p < \infty ,$$

where $f(x) \sim \sum_{1}^{\infty} a_n \cos nx$.

Later on Khan [1] in 1968, obtained several results involving $L(p,\alpha)$ class, which generalise all the above results for cosine series. His results are as follows .

<u>Theorem H</u>. Let $L^{1/p}(1/x)\ f(x) \in L(p,\alpha)$ with $1 \le p < \infty$, $-1 < \alpha p < p-1$, where $f(x) \sim \sum_1^\infty a_n \cos nx$ with $a_n \ge 0$. If $A_n = \sum_{j=[n/2]}^{n} a_j$, then

$$\sum_1^\infty n^{-2-\alpha p}\ L(n)\ A_n^p < \infty ,$$

and

$$\sum_{n=1}^\infty n^{-2-\alpha p}\ L(n)\ A_n^p \le B(\alpha,p) \| L^{1/p}(1/x)\ f(x) \|_{p,\alpha}^p .$$

<u>Theorem I</u>. Let $1 \le p < \infty$, $-1 < \alpha p < p-1$. Suppose that $\{a_n\}$ is a sequence of numbers such that $a_n \to 0$ and

$$\left\{ \sum_1^\infty n^{p-\alpha p-2}\ L(n)\ \left(\sum_{j=n}^\infty |\triangle a_j| \right)^p \right\}^{1/p} < \infty .$$

Then

$$L^{1/p}(1/x)\ f(x) \in L\ (p,\alpha)$$

and

$$\| L^{1/p}(1/x) f(x) \|_{p,\alpha}^p \le B(\alpha,p) \sum_1^\infty n^{p-\alpha p-2}\ L(n)\ \left(\sum_{j=n}^\infty |\triangle a_j| \right)^p ,$$

where $f(x) \sim \sum_1^\infty a_n \cos nx$ and $\triangle a_j = a_j - a_{j+1}$.

<u>Theorem J.</u> Let $\{a_n\}$ be a positive sequence tending to zero and $\{n^{-k} a_n\}$ be monotonically decreasing for some non-negative integer k. If $1 \le p < \infty$, and $-1 < \alpha p < p-1$, then $L^{1/p}(1/x)\, f(x) \in L(p,\alpha)$ iff $\sum_{1}^{\infty} n^{p-\alpha p-2} L(n)\, a_n^p < \infty$, where $f(x) \sim \sum_{1}^{\infty} a_n \cos nx$.

In this chapter we propose to obtain certain generalization of all these results.

4.3 We prove the following theorems.

<u>Theorem 1. Let</u> $\lambda(\pi/x)\, L^{1/p}(1/x) f(x) \in L(p,\alpha)$ <u>with</u> $1 \le p < \infty$, $-1 < \alpha p < p-1$, <u>where</u> $f(x) \sim \sum_{1}^{\infty} a_n \cos nx$ <u>with</u> $a_n \ge 0$. <u>If</u> $A_n = \sum_{j=[n/2]}^{n} a_j$, <u>then</u>

$$\sum_{n=1}^{\infty} n^{-2-\alpha p} L(n)\, \lambda^p(n)\, A_n^p < \infty,$$

<u>and</u>

$$\sum_{1}^{\infty} n^{-2-\alpha p} L(n)\, \lambda^p(n)\, A_n^p \le B(\alpha,p) \| L^{1/p}(1/x)\, \lambda(\pi/x) f(x) \|_{p,\alpha}^p$$

<u>where</u> $\lambda(x)$ <u>is a positive function such that</u>

(4.3.1) $x^{-\alpha+1-1/p-\delta}\, \lambda(x) \uparrow$ <u>as</u> $x \to \infty$ for some small $\delta > 0$.

<u>Theorem 2. Let</u> $1 \leq p < \infty$ and $-1 < \alpha p < p-1$. <u>Suppose that</u> $\{a_n\}$ <u>is a sequence of numbers such that</u> $a_n \to 0$ <u>and</u>

$$\left\{ \sum_{1}^{\infty} n^{p-\alpha p-2} \lambda^p(n)\, L(n) \left(\sum_{j=n}^{\infty} |\Delta a_j| \right)^p \right\}^{1/p} < \infty .$$

<u>Then</u>

$$L^{1/p}(1/x)\, \lambda(\pi/x)\, f(x) \in L(p,\alpha),$$

<u>and</u>

$$\| \lambda(\pi/x)\, L^{1/p}(1/x) f(x) \|_{p,\alpha} \leq B(\alpha,p) \sum_{1}^{\infty} n^{p-\alpha p-2} \lambda^p(n) L(n) \left(\sum_{n}^{\infty} |\Delta a_j|\right)^p$$

<u>where</u> $\lambda(x)$ <u>is a positive function such that</u>

(4.3.P) $\quad x^{-\alpha-1/p+\delta}\, \lambda(x) \downarrow$ <u>as</u> $x \to \infty$ <u>for some small</u> $\delta > 0$,

<u>and</u> $f(x) \sim \sum_{1}^{\infty} a_n \cos nx$.

<u>Theorem 3. Let</u> $\{a_n\}$ <u>be a positive sequence tending to zero and</u> $\{n^{-k} a_n\}$ <u>be monotonic decreasing for some non-negative real number</u> k. <u>If</u> $1 \leq p < \infty$, $-1 < \alpha p < p-1$ <u>and</u> $0 \leq \gamma < \alpha + 1/p$, <u>then</u>

$$x^{-\gamma} L^{1/p}(1/x)\, f(x) \in L(p, \alpha)$$

<u>iff</u>

$$\sum_{n=1}^{\infty} n^{p-\alpha p+\gamma p-2} L(n)\, a_n^p < \infty ,$$

where $f(x) \sim \sum_{1}^{\infty} a_n \cos nx.$

The corresponding theorems for sine series are as follows.

<u>Theorem</u> 4. <u>Let</u> $\lambda(\pi/x)\, L^{1/p}(1/x) g(x) \in L(p,\alpha)$ <u>with</u> $1 \le p < \infty$, $-1-p < \alpha p < p-1$, <u>where</u> $g(x) \sim \sum_{1}^{\infty} a_n \sin nx$ <u>with</u> $a_n \geq 0$. <u>If</u> $A_n = \sum_{j=[\frac{n}{2}]+1}^{n} a_j$, <u>then</u>

$$\sum_{n=1}^{\infty} n^{-2-\alpha p} \lambda^p(n) L(n)\, A_n^p \le B^*(\alpha,p)\, \| L^{1/p}(1/x)\lambda(\pi/x) g(x) \|_{p,\alpha}^{p},$$

<u>where</u> $\lambda(x)$ <u>satisfies the condition</u> (4.3.1).

<u>Theorem</u> 5. <u>Let</u> $1 \le p < \infty$ <u>and</u> $-1-p < \alpha p < p-1$. <u>Suppose that</u> $\{a_n\}$ <u>is a sequence of numbers such that</u> $a_n \to 0$ <u>and</u>

$$\left\{ \sum_{n=1}^{\infty} n^{p-\alpha p-2} \lambda^p(n)\, L(n) \left(\sum_{j=n}^{\infty} |\triangle a_j| \right)^p \right\}^{1/p} < \infty,$$

<u>then</u>

$$L^{1/p}(1/x)\, \lambda(\pi/x)\, g(x) \in L(p,\alpha),$$

<u>and</u>

$$\| L^{1/p}(1/x) \lambda(\pi/x) g(x) \|_{p,\alpha}^{p} \le B(\alpha,p) \sum_{1}^{\infty} n^{p-\alpha p-2} \lambda^p(n) L(n) \left(\sum_{j=n}^{\infty} |\triangle a_j|\right)^p,$$

---

* where B and $B(\alpha,p)$ are constants, not necessarily the same at each occurrence.

where $\lambda(x)$ *is a positive function such that*

(4.3.3) $$x^{-\alpha-1/p-1+\delta}\,\lambda(x)\downarrow \text{ as } x\to\infty$$ *for some small*

$\delta > 0$, *and* $g(x) \sim \sum_{1}^{\infty} a_n \sin nx.$

Theorem 6. Let $\{a_n\}$ *be a positive sequence tending to zero and* $\{n^{-k} a_n\}$ *be monotonic decreasing for some non-negative real number* k . *If* $1 \leq p \leq \infty$, $-1-p < \alpha p < p-1$ *and* $0 \leq \gamma < \alpha+1+1/p$, *then*

$$x^{-\gamma} L^{1/p}\left(\frac{1}{x}\right) g(x) \in L(p,\alpha)$$

iff

$$\sum_{1}^{\infty} n^{p-\alpha p-2+\gamma p} L(n)\, a_n^{p} < \infty,$$

where

$$g(x) \sim \sum_{1}^{\infty} a_n \sin nx.$$

4.4 We require the following lemmas for the proof of our Theorems.

Lemma 1. (Askey and Wainger [1]). *Let* $\{a_n\}$ *be positive and tend to zero and* $\{n^{-k} a_n\}$ *be monotonic decreasing for some* *non-negative real number* k, *then*

$$\sum_{j=n}^{\infty} |\Delta a_j| \leq B \sum_{j=n}^{\infty} \frac{a_j}{j} + a_n ,$$

where B *is some positive constant.**

Lemma 2. (Khan [1]). *Let* $\sum_{n=1}^{\infty} a_n$ *be a series of positive terms such that*

$$\sum_{n=1}^{\infty} n^{-c} L(n) (n a_n)^p < \infty , \quad p \geq 1, \quad c < 1.$$

If

$$A_n = \sum_{\nu = n}^{\infty} a_\nu ,$$

then

$$\sum_{1}^{\infty} n^{-c} L(n) A_n^p < \infty ,$$

and

$$\sum_{1}^{\infty} n^{-c} L(n) A_n^p \leq K \sum_{1}^{\infty} n^{-c} L(n) (n a_n)^p ,$$

where K *is some constant depending on* c *and* p *and* $L(x)$ *is a*

---

* In Lemma 1 the authors have assumed that k should be a non-negative integer but it can be easily verified that the Lemma remains true for all $k \geq 0$.

slowly increasing function in the sense of Karamata [1] .

Lemma 3. Let $f(x) \geq 0$ *for* $x \geq 0$ *and let* $F(x) = \int_0^x f(u)\,du$. *If* $q \geq p \geq 1$, *then*

$$(4.4.1) \quad \left\{ \int_0^\infty t^{-1} \{\mu(t)\, L(1/t) F(t)\}^q \, dt \right\}^{1/q} \leq K \left\{ \int_0^\infty t^{-1} \{t\,\mu(t) L\,(1/t) f(t)\}^p \, dt \right\}^{1/p},$$

and

$$\left\{ \int_0^\infty t^{-1} \{\mu(t)\; L(t)\; F(t)\}^q \, dt \right\}^{1/q} \leq K \left\{ \int_0^\infty t^{-1} \{t\, \mu\,(t)\; L(t)\; f(t)\}^p \, dt \right\}^{1/p},$$

where $\mu(t)$ *is a positive function such that* $t^{\delta}\mu(t)$ *is decreasing as* $t \to \infty$ *for some small* $\delta > 0$ *and* K *is a positive constant.*

It may be remarked that the special case $\mu(t) = t^{-1-\gamma}$, $\gamma > -1$ of this result is due to Igari [1] .

Proof of Lemma 3. First we prove (4.4.1). Consider the case $q \geq p > 1$. Put

$$J = \left[\int_0^\infty t^{-1} \;\{t\, \mu\,(t)\; L(1/t) f(t)\}^p \, dt \right]^{1/p}$$

and let α be a constant such that $\alpha < \frac{1}{p'}$, $\frac{1}{p} + \frac{1}{p'} = 1$.
Applying Hölder's inequality with indices q, p' and $\frac{pq}{q-p}$, we have

$$F(t) = \int_0^t f(u)\ du$$

$$= \int_0^t u^{\alpha - \frac{(p-1)(q-p)}{pq}} (\mu(u)\ L(1/u))^{-\frac{q-p}{q}} f^{p/q}(u)\ u^{-\alpha} \times$$

$$\left\{ u^{p-1} (\mu(u) L(\tfrac{1}{u}) f(u))^p \right\}^{\frac{q-p}{pq}} du$$

$$\leq \left(\int_0^t u^{\alpha q - \frac{(p-1)(q-p)}{p}} (\mu(u) L(\tfrac{1}{u}))^{-(q-p)} f^p(u)\, du \right)^{1/q} \times$$

$$\left(\int_0^t u^{-\alpha p'}\ du\right)^{1/p'} \left(\int_0^t u^{-1} (u\, \mu(u) L(\tfrac{1}{u}) f(u))^p\, du\right)^{\frac{q-p}{pq}}$$

$$\leq K\ t^{1/p' - \alpha}\ J^{1-p/q} \left\{ \int_0^t u^{\alpha q - \frac{(p-1)(q-p)}{p}} (\mu(u) L(\tfrac{1}{u}))^{-q+p} \right.$$

$$\left. \cdot f^p(u)\ du \right\}^{1/q}$$

that is

$$F^q(t) \leq K t^{q/p' - \alpha q}\ J^{q-p} \int_0^t u^{\alpha q - \frac{(p-1)(q-p)}{p}} (\mu(u) L(\tfrac{1}{u}))^{-q+p} f^p(u)\, du.$$

Now, we have

$$t^{-1}(\mu(t)\ L\ (\tfrac{1}{t}))^{q}\ F^{q}(t)$$

$$\leq K\ t^{-1-\alpha q+q/p'}(\mu(t)L(\tfrac{1}{t}))^{q}\ J^{q-p}\int_{0}^{t} u^{\alpha q-\frac{(p-1)(q-p)}{p}}$$

$$(\mu(u)L(\tfrac{1}{u}))^{-q+p}\ f^{p}(u)\,du$$

or

$$\int_{0}^{\infty} t^{-1}\ (\mu(t)\ L\ (\tfrac{1}{t})\ F(t)\ )^{q}\ dt$$

$$\leq K\ J^{q-p}\int_{0}^{\infty} t^{-1-\alpha q+q/p'}(\mu(t)L(\tfrac{1}{t}))^{q}\ (\int_{0}^{t} u^{\alpha q-\frac{(p-1)(q-p)}{p}}\ .$$

$$.\ (\mu(u)L(\tfrac{1}{u}))^{-q+p}\ f^{p}(u)\,du\ )\ dt$$

Changing the order of integration the above expression is

$$= K\ J^{q-p}\int_{0}^{\infty} u^{\alpha q-\frac{(p-1)(q-p)}{p}}(\mu(u)\ L(\tfrac{1}{u}))^{-q+p}\ f^{p}(u)$$

$$(\int_{u}^{\infty} t^{-1-\alpha q+q/p'}(\mu(t)L(\tfrac{1}{t})\ )^{q}\ dt\ )\ du$$

$$= K\ J^{q-p}\int_{0}^{\infty} u^{-1+p}(\mu(u)L(\tfrac{1}{u}))^{-q+p}\ f^{p}(u)\ K(u)\ du$$

where

$$K(u) = u^{\alpha q - q/p'} \int_u^{\infty} t^{-\alpha q + q/p' - 1} \left(\mu(t) L\left(\tfrac{1}{t}\right) \right)^q dt$$

$$\leq \mu^q(u)\, u^{\alpha q - q/p' + q\delta} \int_u^{\infty} t^{-\alpha q + q/p' - 1 - q\delta} L^q\left(\tfrac{1}{t} \right) dt$$

$$= \mu^q(u) \int_0^1 T^{\alpha q + q\delta - q/p' - 1} L^q\left(T/u \right) dT$$

$$\leq K\ \mu^q(u)\, L^q\left(\tfrac{1}{u} \right) \int_0^1 T^{\alpha q + q\delta - q/p' - 1 - \varepsilon q}\, dT$$

$$\leq K\ \mu^q(u)\, L^q\left(\tfrac{1}{u} \right)$$

provided that we first choose $\delta > \frac{1}{p'} - \alpha$ and thus $\varepsilon > 0$ such that $\alpha + \delta - \frac{1}{p'} - \varepsilon > 0$.

Thus we obtain

$$\int_0^{\infty} t^{-1} \left\{ \mu(t)\, L\left(\tfrac{1}{t} \right) F(t) \right\}^q dt$$

$$\leq K\, J^{q-p} \int_0^{\infty} u^{-1+p} \left(\mu(u)\, L\left(\tfrac{1}{u} \right) f(u) \right)^p du$$

$$\leq K\, J^q$$

that is to say

$$\left\{ \int_0^{\infty} t^{-1} \left(\mu(t) L\left(\tfrac{1}{t}\right) F(t) \right)^q dt \right\}^{1/q} \leq K \left\{ \int_0^{\infty} t^{-1} \left(t\, \mu(t) L\left(\tfrac{1}{t}\right) f(t) \right)^p dt \right\}^{1/p}.$$

Thus, we have proved (4.4.1). In the case $q \geq p = 1$, put $\alpha = 0$ and the inequality may be obtained by similar arguements with indices q and q'. The inequality (4.4.2) can be proved in a similar manner.

This completes the proof of Lemma 3.

4.5 <u>Proof of Theorem 1</u>. Let

$$f_1(x) = \int_0^x f(u)\, du\ , \quad f_2(x) = \int_0^x f_1(u)\, du\ ,$$

then

$$f_2(x) = \sum_{j=1}^{\infty} a_j \cdot j^{-2} (1 - \cos jx)$$

$$\geq \sum_{j=[\frac{n}{2}]}^{n} a_j \cdot j^{-2} (1 - \cos jx)$$

$$= 2 \sum_{j=[\frac{n}{2}]}^{n} a_j \cdot j^{-2} \sin^2 \frac{jx}{2}$$

$$\geq 2 \sum_{j=[\frac{n}{2}]}^{n} a_j \cdot j^{-2} \frac{4}{\pi^2} \cdot \frac{j^2 x^2}{4}\ , \quad \pi/n+1 \leq x \leq \pi/n\ ,$$

$$= B\, x^2 \sum_{j=[\frac{n}{2}]}^{n} a_j\ ,$$

where B is some positive constant.

Now

$$\sum_{n=2}^{\infty} n^{-2-\alpha p} L(n) \lambda^{p}(n) A_n^{p}$$

$$\leq B \sum_{n=2}^{\infty} \int_{\pi/n+1}^{\pi/n} x^{\alpha p-2p} L\left(\frac{1}{x}\right) \lambda^{p}(\pi/x) f_2^{p}(x)\, dx$$

$$= B \int_{0}^{\pi/2} x^{\alpha p-2p} L\left(\frac{1}{x}\right) \lambda^{p}(\pi/x) f_2^{p}(x)\, dx$$

$$\leq B \int_{0}^{\pi/2} x^{\alpha p-p} L\left(\frac{1}{x}\right) \lambda^{p}(\pi/x) |f_1(x)|^{p}\, dx$$

$$\leq B \int_{0}^{\pi/2} x^{\alpha p} L\left(\frac{1}{x}\right) \lambda^{p}(\pi/x) |f(x)|^{p}\, dx$$

$$\leq B \int_{0}^{\pi/2} (\sin x)^{\alpha p} L\left(\frac{1}{x}\right) \lambda^{p}(\pi/x) |f(x)|^{p}\, dx$$

$$\leq B \left\| L^{1/p}\left(\frac{1}{x}\right) \lambda(\pi/x) f(x) \right\|_{p,\alpha}^{p}$$

$$< \infty,$$

by virtue of the result (4.4.1) of Lemma 3 and the hypotheses of Theorem 1.

This completes the proof of Theorem 1.

4.6 <u>Proof of Theorem</u> 2. Since we are given that

$$\sum_{n=1}^{\infty} n^{p-\alpha p-2} \lambda^{p}(n)\, L(n) \left(\sum_{j=n}^{\infty} |\Delta a_j| \right)^{p} < \infty ,$$

it follows on putting $n = 1$ that

$$\sum_{j=1}^{\infty} |\Delta a_j| < \infty ,$$

and therefore the fourier series of f converges for $x > 0$ (Zygmund [1]), and we have

$$f(x) = \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \sum_{n=1}^{k} a_n \cos nx + \sum_{n=k+1}^{\infty} a_n \cos nx.$$

Let $D_n(x) = \frac{1}{2} + \sum_{\nu=1}^{n} \cos \nu x$, then

$$\sum_{n=k+1}^{N} a_n \cos nx = \sum_{n=k+1}^{N-1} \Delta a_n \left(\sum_{\nu=0}^{n} \cos \nu x \right) + a_N \sum_{\nu=0}^{N} \cos \nu x$$

$$- a_{k+1} \sum_{\nu=0}^{k} \cos \nu x$$

$$= \sum_{n=k+1}^{N-1} \Delta a_n \left(\tfrac{1}{2} + D_n(x)\right) + a_N\left(\tfrac{1}{2} + D_N(x)\right) - a_{k+1}\left(\tfrac{1}{2} + D_k(x)\right)$$

$$= \sum_{n=k}^{N} \Delta a_n \left(\tfrac{1}{2} + D_n(x) \right) - \Delta a_N \left(\tfrac{1}{2} + D_N(x) \right)$$

$$- \Delta a_k\left(\tfrac{1}{2} + D_k(x)\right) + a_N\left(\tfrac{1}{2} + D_N(x)\right) - a_{k+1}\left(\tfrac{1}{2} + D_k(x)\right)$$

$$= \sum_{n=k}^{N} (\triangle a_n)\; D_n(x) + a_{N+1}\; D_N(x) - a_k\; D_k(x).$$

Now making $N \to \infty$, we have

$$\sum_{n=k+1}^{\infty} a_n \cos nx = \sum_{n=k}^{\infty} (\triangle a_n)\; D_n(x) - a_k\; D_k(x).$$

Hence for any n ,

$$|f(x)| \le \sum_{n=1}^{k} |a_n| + \sum_{n=k}^{\infty} |\triangle a_n|\; |D_n(x)| + |a_k|\; |D_k(x)|$$

Since, $D_n(x) = O(1/x)$, $\pi/n+1 \le x \le \pi/n$, we have

$$|f(x)| \le A_k + O(1/x) \sum_{n=k}^{\infty} |\triangle a_n| + O(1/x)\; |a_k| \;.$$

where

$$A_k = \sum_{n=1}^{k} |a_n| \;.$$

Now,

$$\int_0^{\pi/2} L(\tfrac{1}{x})\; \lambda^p(\pi/x)\; (\sin x)^{\alpha p}\; |f(x)|^p\, dx$$

$$\le B \int_0^{\pi/2} x^{\alpha p}\, L(\tfrac{1}{x})\; \lambda^p\; (\pi/x)\; |f(x)|^p\, dx$$

$$= B \sum_{n=2}^{\infty} \int_{\pi/n+1}^{\pi/n} x^{\alpha p}\, L(\tfrac{1}{x})\; \lambda^p\; (\pi/x)\; |f(x)|^p\, dx$$

$$\leq B \sum_{n=2}^{\infty} \int_{\pi/n+1}^{\pi/n} x^{\alpha p} L(\tfrac{1}{x}) \lambda^{p}(\pi/x) \Big(A_n + \frac{1}{x} \sum_{k=n}^{\infty} |\Delta a_k| + \frac{1}{x} |a_n| \Big)^p dx$$

$$\leq B \sum_{n=2}^{\infty} \int_{\pi/n+1}^{\pi/n} x^{\alpha p} L(\tfrac{1}{x}) \lambda^{p}(\pi/x) A_n^p dx +$$

$$+ B \sum_{n=2}^{\infty} \int_{\pi/n+1}^{\pi/n} x^{\alpha p} L(\tfrac{1}{x}) \lambda^{p}(\pi/x) x^{-p} \Big(\sum_{k=n}^{\infty} |\Delta a_k|\Big)^p dx +$$

$$+ B \sum_{n=2}^{\infty} \int_{\pi/n+1}^{\pi/n} x^{\alpha p} L(\tfrac{1}{x}) \lambda^{p}(\pi/x) x^{-p} |a_n|^p dx$$

$$\leq B \sum_{n=1}^{\infty} n^{-2-\alpha p} L(n) \lambda^{p}(n) A_n^p +$$

$$+ B \sum_{n=2}^{\infty} n^{p-\alpha p-2} L(n) \lambda^{p}(n) \Big(\sum_{k=n}^{\infty} |\Delta a_k|\Big)^p +$$

$$+ B \sum_{n=2}^{\infty} n^{p-\alpha p-2} L(n) \lambda^{p}(n) |a_n|^p$$

$$\leq B \sum_{n=2}^{\infty} n^{-2-\alpha p} L(n) \lambda^{p}(n) A_{n-2}^p$$

$$+ B \sum_{n=2}^{\infty} n^{-2-\alpha p} L(n) \lambda^{p}(n) \big(|a_{n-1}| + |a_n|\big)^p$$

$$+ B \sum_{n=2}^{\infty} n^{p-\alpha p-2} L(n) \lambda^{p}(n) \Big(\sum_{k=n}^{\infty} |\Delta a_k|\Big)^p$$

$$= J_1 + J_2 + J_3 , \text{ say.}$$

Evidently, in view of the hypothesis $J_3 = O(1)$.

Again,

$$J_2 \leq B \sum_{n=2}^{\infty} n^{-2-\alpha p} L(n) \lambda^p(n) \, |a_{n-1}|^p$$

$$+ B \sum_{n=2}^{\infty} n^{-2-\alpha p} L(n) \lambda^p (n) \, |a_n|^p$$

$$\leq B \sum_{n=1}^{\infty} n^{p-\alpha p-2} L(n) \lambda^p(n) \, |a_n|^p$$

$$\leq B \sum_{n=1}^{\infty} n^{p-\alpha p-2} L(n) \lambda^p(n) \left(\sum_{k=n}^{\infty} |\triangle a_k| \right)^p$$

$$= O(1) .$$

Now put $q(x) = |a_n|$ for $n \leq x < n+1$ with $a_o = o$ ($n = 0, 1, 2, \ldots$) and

$$Q(x) = \int_o^x q(t) \, dt ,$$

then we have

$$J_1 \leq B \sum_{n=2}^{\infty} \int_{n-1}^{n} x^{-\alpha p-2} L(x) \lambda^p(x) Q^p(x) \, dx$$

$$= B \int_1^{\infty} x^{-\alpha p-2} L(x) \lambda^p (x) Q^p (x) \, dx .$$

Applying the result (4.4.2) of Lemma 3 (on taking $q=p$ and $\mu(x) = x^{-\alpha-1/p}\lambda(x)$) we observe that the above integral is

$$\leq B \int_{1}^{\infty} x^{-2-\alpha p+p} L(x)\, \lambda^{p}(x)\, q^{p}(x)\, dx$$

$$= B \sum_{n=1}^{\infty} \int_{n}^{n+1} x^{-\alpha p-2+p} L(x)\, \lambda^{p}(x)\, q^{p}(x)\, dx$$

$$\leq B \sum_{n=1}^{\infty} n^{p-\alpha p-2} L(n)\, \lambda^{p}(n)\, |a_n|^{p}$$

$$= O(1).$$

Similarly we can prove

$$\int_{\pi/2}^{\pi} L^{1/p}(1/x)\, \lambda^{p}(\pi/x)\, |f(x)|^{p} \sin^{\alpha p} x\, dx < \infty .$$

Hence,

$$L^{1/p}(1/x)\, \lambda(\pi/x)\, f(x) \in L(p,\alpha),$$

and

$$\| L^{1/p}(1/x)\lambda(\pi/x)f(x) \|^{p}_{p,\alpha} \leq B \sum_{n=1}^{\infty} n^{p-\alpha p-2} L(n)\lambda^{p}(n)\Big(\sum_{k=n}^{\infty} |\Delta a_k|\Big)^{p}.$$

This completes the proof of Theorem 2.

4.7 <u>Proof of Theorem</u> 5. <u>Necessity</u>: Suppose that $x^{-\gamma} L^{1/p}(1/x) f(x) \in L(p,\alpha)$, then we have to prove that

$$\sum_{n=1}^{\infty} n^{p-\alpha p+\gamma p-2} L(n)\, a_n^p < \infty .$$

Since $\{ n^{-k} a_n \}$ is monotonic decreasing, we have

$$a_n = a_n \cdot n^{-k} n^{k} \leq B\, n^{k-1} \sum_{j=[\frac{n}{2}]}^{n} a_j \cdot j^{-k}$$

$$\leq B\, n^{-1} \sum_{j=[\frac{n}{2}]}^{n} a_j .$$

We have therefore by Theorem 1,

$$\sum_{n=1}^{\infty} n^{p-\alpha p+\gamma p-2} L(n)\, a_n^p$$

$$\leq B \sum_{n=1}^{\infty} n^{-\alpha p+\gamma p-2} L(n) \Big(\sum_{j=[\frac{n}{2}]}^{n} a_j \Big)^p$$

$$< \infty .$$

<u>Sufficiency</u> : Now suppose that

$$\sum_{n=1}^{\infty} n^{p-\alpha p+\gamma p-2} L(n)\, a_n^p < \infty .$$

Then

$$\sum_{n=1}^{\infty} n^{p-\alpha p+\gamma p-2} L(n) \left(\sum_{j=n}^{\infty} |\Delta a_j| \right)^p$$

$$\leq B \sum_{n=1}^{\infty} n^{p-\alpha p+\gamma p-2} L(n) \left(\sum_{j=n}^{\infty} \frac{a_j}{j} + a_n \right)^p, \quad \text{(by Lemma 1)}$$

$$\leq B \sum_{n=1}^{\infty} n^{p-\alpha p+\gamma p-2} L(n) \left(\sum_{j=n}^{\infty} \frac{a_j}{j} \right)^p + B \sum_{n=1}^{\infty} n^{p-\alpha p+\gamma p-2} L(n)\, a_n^p$$

$$< \infty, \qquad \text{(by Lemma 2)}$$

and therefore by Theorem 2,

$$L^{1/p}(1/x)\, x^{-\gamma} f(x) \in L\,(p, \alpha).$$

Thus Theorem 3 is proved.

4.8 The proofs of Theorems 4, 5 and 6 are similar to those of Theorems 1, 2 and 3 and hence omitted.

Chapter V

INTEGRABILITY THEOREMS FOR CERTAIN TRIGONOMETRIC SERIES WITH δ-QUASI-MONOTONE COEFFICIENTS

5.1 A sequence $\{a_n\}$ is said to be monotonic decreasing if $a_{n+1} \leq a_n$, $n = 1, 2, \ldots$. It is said to be a null sequence if $a_n \to 0$.

The idea of decreasing null sequence was generalised in the form of a quasi-monotone sequence by Shah [1] and Szász [1] in the following manner.

A sequence $\{a_n\}$ of positive numbers is said to be quasi-monotonic if, and only if , $n^{-\beta} a_n \downarrow 0$ for some $\beta > 0$, or equivalently, if, and only if, $\triangle a_n \geq - \frac{\alpha a_n}{n}$ for some $\alpha > 0$, where $\triangle a_n = a_n - a_{n+1}$.

It is clear that if $\{a_n\}$ is a positive monotonic decreasing sequence then it is also quasi-monotonic. However, the converse need not be true.

The quasi-monotonic sequences are known to share many of the important properties of decreasing sequences. For example , Oliver's theorem, Cauchy's condensation test for convergence and a number of results about trigonometric series have been found to be true for quasi-monotonic sequences.

In 1965, Boas [2] considered a more general definition of quasi-monotonic sequence. According to him a sequence $\{a_n\}$ is said to be δ-quasi-monotonic if $a_n \to 0$; $a_n > 0$ ultimately and $\triangle a_n \geq -\delta_n$ where $\{\delta_n\}$ is a sequence of positive numbers. By taking $\delta_n = \frac{\alpha\, a_n}{n}$ we find that δ-quasi-monotone sequence becomes simply a quasi-monotone sequence.

A number of theorems concerning δ-quasi-monotone sequences was obtained by Boas [2], who restricted δ_n, his various results, by the condition

$$\sum_{n=1}^{\infty} \delta_n n^{\gamma} < \infty \quad (\gamma > 0) \quad \text{or} \quad \sum_{n=1}^{\infty} \delta_n \log n < \infty .$$

A positive function $L(x)$ is said to be " slowly increasing" in the sense of Karamata [1] if it is continuous for $x \geq 0$ and

$$\lim_{x\to\infty} \frac{L(tx)}{L(x)} = 1 \quad \text{for every fixed } t > 0 .$$

Some of the important properties of such a function are as follows*.

---

*P_1, P_2, P_3 and P_4 are due to Karamata [1] and P_5 due to Igari [1].

P_1 : $\frac{L(tx)}{L(x)} \to 1$ as $x \to \infty$, uniformly for $0 < a \leq t \leq b < \infty$.

P_2 : If $f(x) \sim L(x)$, $x \to \infty$, then $f(x)$ is also a slowly increasing function.

P_3 : $x^{\alpha} L(x) \to \infty$, $x^{-\alpha} L(x) \to 0$, as $x \to \infty$ for every $\alpha > 0$.

P_4 : If we write for some $\alpha > 0$,

$$\bar{L}_1(x) = x^{-\alpha} \operatorname*{Max}_{0 \leq t \leq x} \{t^{\alpha} L(t)\}, \quad \underline{L}_1(x) = x^{\alpha} \operatorname*{Min}_{0 < t \leq x} \{t^{-\alpha} L(t)\},$$

$$\bar{L}_2(x) = x^{\alpha} \operatorname*{Max}_{x \leq t < \infty} \{t^{-\alpha} L(t)\}, \quad \underline{L}_2(x) = x^{-\alpha} \operatorname*{Min}_{x \leq t < \infty} \{t^{\alpha} L(t)\},$$

then $\bar{\underline{L}}_k(x) \sim L(x)$ as $x \to \infty$, for $k = 1, 2$.

P_5 : For any $\alpha > 0$, we have

$$L(tu) \leq C_1 t^{-\alpha} L(u) \quad \text{for every } u \geq 0 \text{ and } 0 < t \leq 1,$$

$$L\left(\frac{u}{t}\right) \leq C_2 t^{-\alpha} L(u) \quad \text{for every } u \geq 0,\ 0 < t \leq 1,$$

where C_1 and C_2 are positive constants depending on α and L only.

5.2 Concerning integrability of trigonometric series $f(x) = \sum_{n=1}^{\infty} a_n \cos nx$ and $g(x) = \sum_{n=1}^{\infty} a_n \sin nx$, Boas [1] proved the following results, in which he assumed the sequence of coefficients to be monotonic decreasing and tending to zero .

Theorem A. If $a_n \downarrow 0$ and $0 < \gamma < 1$, then $x^{-\gamma} f(x) \in L(0,\pi)$ if, and only if $\sum_{n=1}^{\infty} n^{\gamma-1} a_n < \infty$.

Theorem B. If $a_n \downarrow 0$ and $0 \leq \gamma \leq 1$, then $x^{-\gamma} g(x) \in L(0,\pi)$ if, and only if $\sum_{n=1}^{\infty} n^{\gamma-1} a_n < \infty$.

Later on, G.Sunouchi [1] proved Theorems A and B by a different method.

Shah [2] obtained the following results concerning the integrability of trigonometric series for quasi-monotone sequences which generalize the above theorems.

Theorem C. Let $\{a_n\}$ be a quasi-monotone sequence.

(i) If $0 < \gamma < 1$, then $\sum_{n=1}^{\infty} n^{\gamma-1} a_n$ is convergent if , and only if $x^{-\gamma} f(x) \in L(0,\pi)$.

(ii) $0 < \gamma \leq 1$, then $\sum_{n=1}^{\infty} n^{\gamma-1} a_n$ is convergent if, and only if $x^{-\gamma} g(x) \in L(0,\pi)$.

This theorem was subsequently extended by Boas [2] for

δ-quasi-monotone sequences. His results are as follows :

<u>Theorem</u> D. Let $0 < \gamma < 1$, and let $\{a_n\}$ be a δ-quasi-monotone sequence with $\sum_{n=1}^{\infty} n^{\gamma} \delta_n < \infty$. Then $\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ converges (except perhaps at integral multiples of 2π) to $f(x)$ and $\sum_{n=1}^{\infty} n^{\gamma-1} a_n$ converges if, and only if $x^{-\gamma} f(x) \in L(0,\pi)$.

<u>Theorem</u> E. Let $0 < \gamma \leq 1$, and let $\{a_n\}$ be a δ-quasi-monotone sequence with $\sum_{n=1}^{\infty} n^{\gamma} \delta_n < \infty$. Then $\sum_{n=1}^{\infty} a_n \sin nx$ converges to $g(x)$ and $\sum_{n=1}^{\infty} n^{\gamma-1} a_n$ converges if, and only if $x^{-\gamma} g(x) \in L(0,\pi)$.

<u>Remark</u> :- Suppose that $\{a_n\}$ is a positive sequence tending to zero and $\delta_n = \frac{\alpha\, a_n}{n}$. Let $\sum_{n=1}^{\infty} n^{\gamma-1} a_n < \infty$, and $0 < \gamma < 1$. Then Theorem D asserts that $\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ converges (except perhaps at integral multiples of 2π) to $f(x)$ and $\sum_{n=1}^{\infty} n^{\gamma-1} a_n$ converges iff $x^{-\gamma} f(x) \in L(0,\pi)$. Considering the case $\int - \Sigma$ we observe that Boas had already assumed what he wished to prove. Thus his results (Theorem D and E) suffer from this serious defect. Of course, he could have avoided this by

(say in Theorem D) assuming the convergence of $\frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos nx$ instead of that of $\sum_{n=1}^{\infty} n^{\gamma} \delta_n$ in the part $\int \to \Sigma$.

Aljančić , Bojanić and Tomić [2] , in 1955, generalized Theorems A and B in a different direction. They proved, among others, the following results.

<u>Theorem</u> F. If $0 < \gamma < 1$, $a_n \downarrow 0$, then $x^{-\gamma} L(1/x)\, f(x) \in L(0,\pi)$, if, and only if $\sum_{n=1}^{\infty} n^{\gamma-1} L(n)\, a_n < \infty$.

<u>Theorem</u> G. If $0 < \gamma < 2$, $a_n \downarrow 0$, then $x^{-\gamma} L(1/x) g(x) \in L(0,\pi)$ if, and only if, $\sum_{n=1}^{\infty} n^{\gamma-1} L(n)\, a_n < \infty$.

These results were extended by Yong [1] in 1965 to quasi-monotone sequences in the following form.

<u>Theorem</u> H. Let $\{a_n\}$ be a quasi-monotone sequence with $a_n \to 0$ as $n \to \infty$ and $0 < \gamma < 1$. Then $\sum_{1}^{\infty} n^{\gamma-1} L(n)\, a_n$ converges if, and only if, $\frac{1}{2} a_0 + \sum_{1}^{\infty} a_n \cos nx$ converges everywhere to $f(x)$, save possibly $x = 0$, and $x^{-\gamma} L(1/x)\, f(x) \in L(0,\pi)$.

<u>Theorem</u> I. Let $\{a_n\}$ be a quasi-monotone sequence with $a_n \to 0$, as $n \to \infty$.

(i) For $0 < \gamma < 2$, if $\sum_{1}^{\infty} n^{\gamma-1} L(n) a_n$ converges, then $\sum_{1}^{\infty} a_n \sin nx$ converges everywhere to $g(x)$ and $x^{-\gamma} L(1/x) g(x) \in L(0,\pi)$.

(ii) For $0 < \gamma < 1$, if $\sum_{1}^{\infty} a_n \sin nx$ converges everywhere to $g(x)$ and $x^{-\gamma} L(1/x) g(x) \in L(0,\pi)$, then $\sum_{n=1}^{\infty} n^{\gamma-1} L(n) a_n$ converges.

It may be remarked that if we examine the proof of $\int - \Sigma$ in Theorems H and I, we find that it is sufficient to assume that $\{a_n\}$ is only a positive sequence.

In the present chapter our object is to generalise all the results stated above. In what follows, we prove the following theorems.

Theorem 1. Let $\{a_n\}$ be a δ-quasi-monotone sequence and $0 < \gamma < 1$. If

$$(5.2.1) \qquad \sum_{n=1}^{\infty} n^{\gamma} L(n) \delta_n < \infty,$$

and

$$(5.2.2) \qquad \sum_{n=1}^{\infty} n^{\gamma-1} L(n) a_n \text{ converges, then,}$$

$$(5.2.3) \qquad \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx \text{ converges everywhere}$$

to $f(x)$ *except possibly at* $x=0$ *and*

$$(5.2.4) \qquad x^{-\gamma} L(1/x)\ f(x) \in L(0,\pi).$$

Conversely, if $\{a_n\}$ *is any sequence which is ultimately positive such that* (5.2.3) *holds and if* (5.2.4) *holds, then* (5.2.2) *holds.*

Theorem 2. (i) *Let* $\{a_n\}$ *be a* δ*-quasi-monotone sequence and* $0 < \gamma < 2$. If the series $\sum_1^\infty n^\gamma L(n)\ \delta_n$ *and* $\sum_1^\infty n^{\gamma-1} L(n) a_n$ *are convergent, then* $\sum_1^\infty a_n \sin nx$ *converges everywhere to* $g(x)$ *and* $x^{-\gamma} L(1/x)\ g(x) \in L(0,\pi)$.

(ii) *For* $0 < \gamma < 1$, *if* $\sum_1^\infty a_n \sin nx$ *converges everywhere to* $g(x)$ *and* $x^{-\gamma} L(1/x)\ g(x) \in L(0,\pi)$, *then* $\sum_1^\infty n^{\gamma-1} L(n)\ a_n$ *converges, where* $\{a_n\}$ *is any sequence which is ultimately positive.*

We follow Yong [1] in the proof of our theorems.

5.3 We require the following lemmas for the proof of our theorems.

Lemma 1. (Aljančić, Bojanić and Tomić [2]). *For* $\gamma > 0$,

$$0 < A_1\, n^\gamma L(n) \le \sum_{k=1}^{n} k^{\gamma-1} L(k) \le A_2\, n^\gamma L(n) \ ,$$

where A_1 *and* A_2 *are positive constants.*

Lemma 2. (Aljančić, Bojanić and Tomić [1]*). *Let* α *and* β *be positive. If the integral*

$$\int_{+0}^{\infty} y^{k} \, |f(y)| \, dy \quad < \infty ,$$

for $-\alpha < k < \beta$, *then*

$$\int_{+0}^{\infty} f(y) \, L\,(\lambda y) \, dy \sim L(\lambda) \int_{+0}^{\infty} f(y)\,dy \,, \quad \lambda \to \infty .$$

Lemma 3. If $\{a_n\}$ *is a* δ*-quasi-monotone sequence with* $\sum_{n=1}^{\infty} n^{\gamma} L(n)\, \delta_n < \infty$, $\gamma > 0$ *and* $\sum_{n=1}^{\infty} n^{\gamma-1} L(n)\, a_n$ *converges, then* $\sum_{n=1}^{\infty} n^{\gamma} L(n) \, |\triangle a_n| < \infty$. *Conversely, if* $a_n \to 0$ *and* $\sum_{n=1}^{\infty} n^{\gamma} L(n) \, |\triangle a_n| < \infty$, *then* $\sum_{1}^{\infty} n^{\gamma-1} L(n)\, a_n$ *is convergent.*

Taking $L(x) = 1$ we get a result of Boas [2] while for $\delta_n = \dfrac{c\, a_n}{n}$ we have the corresponding result of Yong [1] .

Proof of Lemma 3. Since $\triangle a_n \geq -\delta_n$, we have

$$|\triangle a_n| \leq \triangle a_n + 2\,\delta_n .$$

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* The authors use in their paper the asymptotic relation $\int_{c/\lambda}^{\infty} f(x) L(\lambda x)\, dx \sim L(\lambda) \int_{+0}^{\infty} f(x)\, dx$ for a fixed $c > 0$.

First suppose that $\sum_{n=1}^{\infty} n^{\gamma-1} L(n)\, a_n$ converges. Then, by Lemma 1, we have

$$\sum_{k=1}^{\infty} k^{\gamma} L(k)\, |\Delta a_k| \le A^{*} \sum_{k=1}^{\infty} |\Delta a_k| \sum_{\nu=1}^{k} \nu^{\gamma-1} L(\nu)$$

$$\le A \sum_{k=1}^{\infty} (\Delta a_k + 2\delta_k) \sum_{\nu=1}^{k} \nu^{\gamma-1} L(\nu)$$

$$= A \sum_{\nu=1}^{\infty} \nu^{\gamma-1} L(\nu) \sum_{k=\nu}^{\infty} (\Delta a_k + 2\delta_k)$$

$$= A \sum_{\nu=1}^{\infty} \nu^{\gamma-1} L(\nu) \left(a_\nu + 2 \sum_{k=}^{\infty} \delta_k \right)$$

$$\le A \sum_{\nu=1}^{\infty} \nu^{\gamma-1} L(\nu)\, |a_\nu| + 2A \sum_{k=1}^{\infty} \delta_k \sum_{\nu=1}^{k} \nu^{\gamma-1} L(\nu)$$

$$\le A \sum_{\nu=1}^{\infty} \nu^{\gamma-1} L(\nu)\, |a_\nu| + A \sum_{k=1}^{\infty} k^{\gamma} L(k)\, \delta_k$$

Now suppose that $\sum_{k=1}^{\infty} k^{\gamma} L(k)\, |\Delta a_k|$ converges. Then, by Abel's transformation, we have

$$\sum_{k=1}^{n+1} k^{\gamma-1} L(k)\, a_k = \sum_{k=1}^{n} \Delta a_k \sum_{\nu=1}^{k} \nu^{\gamma-1} L(\nu) + a_{n+1} \sum_{\nu=1}^{n+1} \nu^{\gamma-1} L(\nu)$$

---

* where A is a constant not necessarily the same at occurrence.

$$\leq \sum_{k=1}^{n} |\triangle a_k| \sum_{\nu=1}^{k} \nu^{\gamma-1} L(\nu) + a_{n+1} \sum_{\nu=1}^{n+1} \nu^{\gamma-1} L(\nu)$$

$$\leq A \sum_{k=1}^{n} |\triangle a_k| \, k^{\gamma} L(k) + a_{n+1} \sum_{\nu=1}^{n+1} \nu^{\gamma-1} L(\nu).$$

Now

$$\left| a_{n+1} \sum_{\nu=1}^{n+1} \nu^{\gamma-1} L(\nu) \right| = \left| \sum_{\nu=1}^{n+1} \nu^{\gamma-1} L(\nu) \sum_{k=n+1}^{\infty} \triangle a_k \right|$$

$$\leq \sum_{\nu=1}^{n+1} \nu^{\gamma-1} L(\nu) \sum_{k=n+1}^{\infty} |\triangle a_k|$$

$$\leq \sum_{k=n+1}^{\infty} |\triangle a_k| \sum_{\nu=1}^{k} \nu^{\gamma-1} L(\nu)$$

$$\leq A \sum_{k=n+1}^{\infty} k^{\gamma} L(k) \, |\triangle a_k|$$

$\to 0$, $n \to \infty$, by virtue of the hypotheses.

Hence, $\sum_{k=1}^{\infty} k^{\gamma-1} L(k) \, a_k$ converges.

This proves Lemma 3.

5.4 <u>Proof of Theorem 1</u>. Proof of $\Sigma \to \int$: By virtue of the hypothesis $\triangle a_n \geq -\delta_n$, we have

$$|\triangle a_n| \leq \triangle a_n + 2 \delta_n .$$

Also the convergence of the series $\sum_{n=1}^{\infty} n^{\gamma} L(n) \, \delta_n$ implies that $\sum_{n=1}^{\infty} \delta_n < \infty$. Therefore, using the condition that $a_n \to 0$, we have

$$\sum_{n=1}^{\infty} |\triangle a_n| \leq \sum_{n=1}^{\infty} \triangle a_n + 2 \sum_{n=1}^{\infty} \delta_n < \infty .$$

Thus $\frac{1}{2} a_o + \sum_{n=1}^{\infty} a_n \cos nx$ converges to $f(x)$ for all x except possibly $x = 0$.

By Abel's transformation, we have

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} (a_n - a_{n+1}) \left(\sum_{k=1}^{n} \cos kx \right)$$

$$= \frac{a_o}{2} + \sum_{n=1}^{\infty} (a_n - a_{n+1}) \left(\frac{\sin(n+\frac{1}{2})x - \sin {}^x/_2}{2 \sin {}^x/_2} \right)$$

$$= \frac{a_o}{2} + \sum_{n=1}^{\infty} (a_n - a_{n+1}) \frac{\cos(n+1){}^x/_2 \sin \frac{nx}{2}}{\sin {}^x/_2}$$

Since $\left(\frac{1}{x} \right)^{-\alpha} L\left({}^1/x \right) \to 0$, as $x \to 0$, it is easy to see that $x^{-\gamma} L\left({}^1/x \right) \in L(0,\pi)$ for $0 < \gamma < 1$.

Therefore, it is sufficient to prove that

$$\int_0^{\pi} x^{-\gamma-1} L({}^1/x) \left| \sum_{n=1}^{\infty} (a_n - a_{n+1}) \cos \left(\frac{n+1}{2}\right) x \sin \frac{nx}{2} \right| dx < \infty.$$

Now

$$\int_0^{\pi} x^{-\gamma-1} L({}^1/x) \left| \sum_{n=1}^{\infty} (a_n - a_{n+1}) \cos\left(\frac{n+1}{2}\right)x \sin \frac{nx}{2} \right| dx$$

$$\leq A \int_0^\pi x^{-\gamma-1} L(1/x) \sum_{n=1}^{\infty} |\triangle a_n| \; |\sin \frac{nx}{2} \cos(\frac{n+1}{2}) x| dx$$

$$\leq A \sum_{n=1}^{\infty} |\triangle a_n| \int_0^\pi x^{-\gamma-1} L(1/x) |\sin \frac{nx}{2}| dx$$

$$(5.4.1) = A \sum_{n=1}^{\infty} |\triangle a_n| n^\gamma L(n) K_n ,$$

where

$$K_n = \frac{1}{n^\gamma L(n)} \int_0^\pi x^{-\gamma-1} L(1/x) \; |\sin \frac{nx}{2}| dx.$$

Putting $y = \frac{1}{nx}$, we have

$$K_n = \frac{1}{L(n)} \int_{1/n\pi}^{\infty} y^{\gamma-1} L(n\, y) \; |\sin \frac{1}{2y}| \, dy$$

$$\sim \frac{L(n)}{L(n)} \int_{+0}^{\infty} y^{\gamma-1} \; |\sin \frac{1}{2y}| \, dy , \; n \to \infty, \quad \text{by Lemma 2,}$$

$$\leq \int_{+0}^{1} y^{\gamma-1} dy + A \int_1^\infty y^{\gamma-2} dy$$

$$< \infty .$$

Hence the expression in (5.4.1) is

$$\leq A \sum_{n=1}^{\infty} n^\gamma L(n) \; |\triangle a_n| < \infty ,$$

by virtue of Lemma 3 and the hypotheses.

Therefore,

$$x^{-\gamma} L(1/x)\ f(x) \in L(0,\pi).$$

Proof of $\int \to \Sigma$: By assumption $\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ converges for $x > 0$. The condition

$$x^{-\gamma} L(1/x) f(x) \in L(0,\pi) \Rightarrow f(x) \in L(0,\pi),$$

and hence a_n is the Fourier coefficient of $f(x)$. Writing $p = [\frac{1}{\delta}]$ and $q = [\frac{1}{x}]$, where $0 < \delta < \pi$, we have

$$p \leq q \quad \text{for} \quad 0 \leq x \leq \delta ;$$

and

$$q \leq p \quad \text{for} \quad \delta \leq x \leq \pi .$$

Supposing, $a_n > 0$ for $n > p_0$, then

$$\sum_{n=p_0}^{p} n^{\gamma-1} L(n) a_n \leq \sum_{n=p_0}^{p} \max_{n \leq k < \infty} \{ k^{\gamma-1} L(k) \} a_n$$

$$= \sum_{n=p_0}^{p} n^{\gamma-1} \bar{L}_2(n)\ a_n$$

$$= \sum_{n=p_0}^{p} n^{\gamma-1} \bar{L}_2(n) \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\ dx$$

$$= \frac{2}{\pi} \int_0^{\pi} f(x) \sum_{n=p_0}^{p} n^{\gamma-1} \bar{L}_2(n) \cos nx\ dx$$

$$\leq \frac{2}{\pi} \int_0^{\pi} |f(x)| \left| \sum_{n=p_0}^{p} n^{\gamma-1} \bar{L}_2(n) \cos nx \right| dx$$

$$= A \left(\int_0^{\delta} + \int_{\delta}^{\pi} \right) = I_1 + I_2, \text{ say.}$$

$$I_1 \leq A \int_0^{\delta} |f(x)| \left(\sum_{n=p_0}^{p} n^{\gamma-1} \bar{L}_2(n) \right) dx$$

$$\leq A \int_0^{\delta} |f(x)| \; q^{\gamma} \bar{L}_2(q) \; dx$$

$$\leq A \int_0^{\delta} x^{-\gamma} L(1/x) \; |f(x)| \; dx$$

$$< \infty .$$

Also

$$I_2 \leq A \int_{\delta}^{\pi} |f(x)| \left(\sum_{n=p_0}^{q} n^{\gamma-1} \bar{L}_2(n) \; dx + A \int_{\delta}^{\pi} |f(x)| \right. .$$

$$\left| \sum_{n=q+1}^{p} n^{\gamma-1} \bar{L}_2(n) \cos nx \right| dx$$

$$\leq A \int_{\delta}^{\pi} x^{-\gamma} L(1/x) \, |f(x)| \, dx + A \int_{\delta}^{\pi} |f(x)| q^{\gamma-1} \bar{L}_2(q) \; .$$

$$\operatorname{Max} \left| \sum_{q}^{p+1} \cos nx \right| dx$$

$$\leq A \int_{\delta}^{\pi} x^{-\gamma} L(1/x) \, |f(x)| \, dx + A \int_{\delta}^{\pi} |f(x)| \, q^{\gamma-1} \bar{L}_2(q) \frac{1}{\sin x/2} \, dx$$

$$\leq A \int_{\delta}^{\pi} x^{-\gamma} L(1/x) \; |f(x)| \, dx$$

$$< \infty .$$

This proves $\int \to \Sigma$ part of Theorem 1.

Thus the Theorem 1 is proved.

5.5 <u>Proof of Theorem</u> 2. (i) <u>Proof of</u> $\Sigma \to \int$:

Since $\Delta a_n \geq -\delta_n$, we have $|\Delta a_n| \leq \Delta a_n + 2\delta_n$.
The convergence of the series $\sum_{n=1}^{\infty} n^{\gamma} L(n)\, \delta_n$ implies that $\sum_{n=1}^{\infty} \delta_n < \infty$. Therefore, by using the condition that $a_n \to 0$, we have

$$\sum_{n=1}^{\infty} |\Delta a_n| \leq \sum_{n=1}^{\infty} \Delta a_n + 2 \sum_{n=1}^{\infty} \delta_n < \infty .$$

Thus, $\sum_{n=1}^{\infty} a_n \sin nx$ converges to $g(x)$ for every x.

Using Abel's transformation, we have

$$g(x) = \frac{1}{2 \sin \frac{x}{2}} \sum_{n=1}^{\infty} \left(\cos \frac{x}{2} - \cos \left(n + \frac{1}{2}\right) x \right) (a_n - a_{n+1})$$

$$= -\frac{1}{2} \tan \frac{x}{4} \sum_{n=1}^{\infty} \Delta a_n + \frac{1}{2\sin \frac{x}{2}} \sum_{n=1}^{\infty} \Delta a_n \left(1 - \cos\left(n + \frac{1}{2}\right) x \right)$$

and hence

$$|g(x)| \leq \frac{1}{2}\tan \frac{x}{4} \sum_{n=1}^{\infty} |\Delta a_n| + \frac{x}{2\sin \frac{x}{2}}\, G(x) ,$$

where $G(x) = \frac{1}{x} \sum_{1}^{\infty} |\Delta a_n| \;(1 - \cos (n + \frac{1}{2})\; x\;).$

Since $(\frac{1}{x})^{-\alpha} L(\frac{1}{x}) \to 0$, as $x \to 0$, it is easy to see that $x^{-\gamma} L(\frac{1}{x}) \tan \frac{x}{4} \in L(0,\pi)$. for $0 < \gamma < 2$.

Now

(5.5.1) $\int_0^{\pi} x^{-\gamma} L(\frac{1}{x})\; G(x)\; dx$

$$= \int_0^{\pi} x^{-\gamma-1} L(\frac{1}{x}) \sum_{n=1}^{\infty} |\Delta a_n| \;(1-\cos(n+\frac{1}{2})\; x\;)\; dx$$

$$= \sum_{n=1}^{\infty} n^{\gamma} L(n)\; |\Delta a_n|\; I_n \;,$$

where

$$I_n = \frac{1}{n^{\gamma} L(n)} \int_0^{\pi} x^{-\gamma-1} L(\frac{1}{x}) \;\left(1-\cos\;(n+\frac{1}{2})x\right) dx$$

$$= \frac{2}{n^{\gamma} L(n)} \int_0^{\pi} x^{-\gamma-1} L(\frac{1}{x})\; \sin^2 (n+\frac{1}{2}) \frac{x}{2}\; dx.$$

Putting $y = \dfrac{1}{(n+\frac{1}{2})x}$, we have

$$I_n = \frac{2(n+\frac{1}{2})^{\gamma}}{n^{\gamma} L(n)} \int_{1/(n+\frac{1}{2})\pi}^{\infty} y^{\gamma-1} \sin^2 \frac{1}{2y}\; L((n+\frac{1}{2})\; y\;)\; dy$$

$$\sim \frac{2}{L(n)} \int_{1/(n+\frac{1}{2})x}^{\infty} y^{\gamma-1} L((n+\tfrac{1}{2})y)\ \sin^2 \frac{1}{2y}\, dy$$

$$\sim 2 \int_{+0}^{\infty} y^{\gamma-1} \sin^2 \frac{1}{2y}\, dy\ , \quad \text{as } n \to \infty, \text{ by Lemma 2.}$$

Then, as $n \to \infty$,

$$I_n \to 2 \int_{+0}^{\infty} y^{\gamma-1} \sin^2 \frac{1}{2y}\, dy < \infty, \quad 0 < \gamma < 2.$$

Hence the expression in (8.5.1) is

$$\leq A \sum_{n=1}^{\infty} n^{\gamma} L(n)\ |\triangle a_n| < \infty ,$$

by Lemma 3 and the hypotheses of Theorem 2.

Thus we have

$$\int_0^{\pi} x^{-\gamma} L(\tfrac{1}{x})\ |g(x)|\ dx \leq A + A \int_0^{\pi} x^{-\gamma} L(\tfrac{1}{x}) G(x)\, dx$$
$$< \infty .$$

This proves $\Sigma - \int$ part of Theorem 2.

(ii) <u>Proof of</u> $\int - \Sigma$. The proof of this part is similar to that of Theorem 1 and hence omitted.

Chapter VI

ON SOME INEQUALITIES FOR FOURIER SERIES

6.1 A non-decreasing continuous real-valued function Φ defined on the non-negative half line and vanishing only at the origin will be called an Orlicz Function (OF). Function $\Phi \in$ OF is said to satisfy Δ_α $(\alpha > 0)$ condition for large u if there are constants $C > 0$ and $u_0 \geq 0$ such that $\Phi(\alpha u) \leq C\ \Phi(u)$, $u \geq u_0$ for every $\alpha > 1$. A convex Orlicz function Φ satisfying the conditions

$$\lim_{u \to 0} \frac{\Phi(u)}{u} = 0 \quad \text{and} \quad \lim_{u \to \infty} \frac{\Phi(u)}{u} = \infty$$

Young function (YF). Function Φ belongs to YF iff it admits a representation

$$\Phi(u) = \int_0^u \phi(x)\, dx ,$$

where $\phi(x)$ $(x \geq 0)$ is positive, $\phi(0) = 0$, continuous on the right, non-decreasing and $\lim_{x \to \infty} \phi(x) = \infty$. We have for such functions the relation

(6.1.1) $$\frac{\Phi(u)}{u} \leq \phi(u) \leq \frac{\Phi(2u)}{u}.$$

We denote by M the class of Orlicz functions Φ which satisfy the following condition of Mulholland [1] .

" There exist a convex function Λ , $\lambda > 1$ and $0<k<1$, such that the inequality

$$\Lambda(u) \leq \Phi^{k}(u) \leq \lambda\,\Lambda(u) \text{ holds for all } u ."$$

We write

$$G(x) = \int_{x/2}^{x} \frac{f(t)}{t}\,dt.$$

6.2 M. Izumi and S.Izumi [1] in 1968, proved among others, the following theorems.

Theorem A. Let $p > 1$ and $s > -1$ and let f be a non-negative, non-increasing and integrable function on $(0,\pi)$. If $x^{s} f^{p}(x)$ is integrable, then we have

$$\int_{0}^{\pi} x^{s}\, G^{p}(x)\, dx \leq A^{*}\int_{0}^{\pi} x^{s}\left(f\left(\tfrac{x}{2}\right)-f(x)\right)^{p}dx + A\left(\int_{\pi/2}^{\pi} f(x)\,dx\right)^{p}.$$

Theorem B. Let $p > 1$ and $s < -1$ and let f be a non-negative and integrable on $(0,\pi)$. If $x^{s} f^{p}(x)$ is integrable , then

*where A denotes a constant, not necessarily the same at each occurrence.

$$\int_0^\pi x^s G^p(x)\, dx \le \left(\frac{p}{-s-1}\right)^p \int_0^\pi x^s \left| f\left(\frac{x}{2}\right) - f(x) \right|^p dx.$$

Theorem C. Let $p > 1$ and $s \neq 1$ and let f be a non-negative, non-constant and integrable on $(0, \infty)$. If

$$x^s f^p(x) \in L(0,\pi) ,$$

then

$$\int_0^\infty x^s G^p(x)\, dx \le \left|\frac{p}{-s-1}\right|^p \int_0^\infty x^s \left| f\left(\frac{x}{2}\right) - f(x) \right|^p dx.$$

Theorem D. Let $p > 1$ and $\{a_n\}$ be a monotonic non-increasing sequence tending to zero and

$$\frac{1}{n} \sum_{m=0}^{n-1} a_m \le 2 \sum_{m=n}^{2n}{}^{*} a_m \quad \text{for all } n \ge 1 ,$$

where Σ^* denotes the sum whose first and last terms are halved. Suppose that

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = \sum_{n=0}^{\infty}{}^{*} a_n \cos nx$$

and that $f(x)$ is non-negative.

(I) If f is L^p integrable and decreases monotonically, then

$$\sum_{n=1}^{\infty} n^{p-2} a_n^p \le A \int_0^\pi \left(f\left(\frac{x}{2}\right) - f(x)\right)^p dx + A\left(\int_{\pi/2}^{\pi} f(x)\, dx\right)^p.$$

(II) More generally, if $-1 < s < p-1$ and $x^s f^p(x)$ is integrable on $(0,\pi)$, then

$$\sum_{n=1}^{\infty} n^{p-s-2} a_n^p$$
$$\leq A \int_0^{\pi} x^s \left| f(\tfrac{x}{2}) - f(x) \right|^p dx + A \frac{\left(\int_{\pi/2}^{\pi} f(x)\,dx\right)^{p^2}}{\left\{\int_0^{\pi/2} f(x)\,dx\right\}^{p^2-p}}$$

(III) If $s < -1$ and $x^s f^p(x)$ is integrable on, $(0,\pi)$, then

$$\sum_{n=1}^{\infty} a_n^p n^{p-s-2} \leq A \int_0^{\pi} x^s \left| f(x) - f(\tfrac{x}{2}) \right|^p dx.$$

The object of this chapter is to obtain certain generalizations of the above theorems. In our theorems we replace the special class L^p of functions by a more general class $L(\Phi)$ where Φ satisfies certain properties. In Theorem 1 we have shown that condition of monotonicity in Theorem A is redundant . A similar remark is also applicable to Theorem D (part I).

6.3 We prove the following theorems.

<u>Theorem 1.</u> <u>Let</u> $\Phi \in \Delta_\alpha \cap YF$ <u>and</u> f <u>be a non-negative and integrable on</u> $(0,\pi)$. <u>If</u> $x^s \Phi(f(x))$ <u>is integrable and</u> $s > -1$, <u>then we have</u>

$$\int_0^\pi x^s \Phi(G(x))\,dx$$

$$\leq A \int_0^\pi x^s \Phi(|f(x)-f(\tfrac{x}{2})|)\,dx + A\,\Phi\left(\int_{\pi/2}^\pi f(x)\,dx\right).$$

__Theorem 2. Let__ $\Phi \in \Delta_\alpha \cap YF$ __and__ f __be non-negative. If__ $x^s \Phi(f(x))$ __is integrable and__ $s < -1$, __then__

$$\int_0^\pi x^s \Phi(G(x))\,dx \leq A \int_0^\pi x^s \Phi(|f(x)-f(\tfrac{x}{2})|)\,dx.$$

__Theorem 3. Let__ $\Phi \in \Delta_\alpha \cap YF$, $s \neq -1$ __and__ f __be a non-negative, non-constant and integrable on__ $(0, \infty)$. __If__ $x^s \Phi(f(x)) \in L(0,\infty)$, __then__

$$\int_0^\infty x^s \Phi(G(x))\,dx \leq A \int_0^\infty x^s \Phi(|f(\tfrac{x}{2}) - f(x)|)\,dx.$$

__Theorem 4. Let__ $\Phi \in \Delta_\alpha \cap YF \cap M$. __Let__ $\{a_n\}$ satisfy __the conditions of Theorem__ D. __Suppose that__

$$f(x) \sim \frac{1}{2} a_0 + \sum_1^\infty a_n \cos nx = \sum_{n=0}^{\infty}{}^{*} a_n \cos nx$$

__and that__ $f(x)$ __is non-negative.__

(i) __If__ $x^s \Phi(f(x))$ __is integrable and__ $-1 < s < \beta-1$, $\beta > 1$, __then__

$$\sum_{n=1}^\infty n^{-s-2} \Phi(n\,a_n)$$

$$\leq A \int_0^\pi x^s \Phi(|f(x)-f(\tfrac{x}{2})|)\,dx + A\,\Phi\left(\int_{\pi/2}^\pi f(x)\,dx\right).$$

(ii) *If* $s < -1$ *and* $x^s \Phi(f(x))$ *is integrable on* $(0,\pi)$, *then*

$$\sum_{n=1}^{\infty} n^{-s-2} \Phi(n a_n) \leq A \int_0^{\pi} x^s \Phi(|f(x) - f(\tfrac{x}{2})|)\, dx.$$

6.4 We require the following lemmas for the proof of our theorems.

Lemma 1. (Izumi M. and Izumi S. [1]). *Let* $\{a_n\}$ *be a monotonic decreasing sequence tending to zero and*

$$\frac{1}{n} \sum_{m=0}^{n-1} a_m \leq 2 \sum_{m=n}^{2n}{}^{*} a_m$$

for all $n \geq 1$, *where* Σ^* *denotes the sum whose first and last terms are halved. If* $f(x) \sim \sum_{n=0}^{\infty}{}^{*} a_n \cos nx$ *is non-negative and integrable, then the* n-th *Fourier coefficient* γ_n *of the even function* Q *defined by*

$$Q(x) = \int_{x/2}^{x} \frac{f(t)}{2 \tan t/2}\, dt, \qquad 0 < x < \pi,$$

is non-negative and further

$$: 0 < \theta < 1, \quad \gamma_n \geq \frac{\theta}{n} \sum_{m=n}^{2n}{}^{*} a_m$$

for all $n \geq 1$.

Lemma 2. (Zygmund [1]). Let $f(x)$ be a non-negative function defined for $x \geq 0$, and let $r > 1$, $s < r-1$. Then, if

$$x^s f^r(x) \in L(0,\infty)$$

so is

$$x^s \left(\frac{F(x)}{x}\right)^r,$$

where

$$F(x) = \int_0^x f(t)\, dt.$$

Moreover,

$$\int_0^\infty x^s \left(\frac{F(x)}{x}\right)^r dx \leq \left(\frac{r}{r-s-1}\right)^r \int_0^\infty x^s f^r(x)\, dx.$$

Lemma 3.† If $\Phi \in F$, $s < \alpha-1$, $\alpha > 1$, and $x^s \Phi(f(x)) \in L(0,\infty)$, then

$$\int_0^\infty \Phi\left(\frac{F(x)}{x}\right) x^s\, dx \leq A \int_0^\infty \Phi(f(x))\, x^s\, dx,$$

where

$$F(x) = \int_0^x f(t)\, dt$$

and $f(x) \geq 0$ for $x \geq 0$.

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† Woyczynski [1] proved this lemma for $s \leq 0$ only. Following the same technique here we extend the result to the case $s < \alpha-1$, $\alpha > 1$.

Proof of Lemma 3. By virtue of the assumption and the Jensen's inequality, there exist an $\alpha > 1$ and a convex function $\Lambda(x)$ such that

$$\Phi^{1/\alpha}\left(\frac{F(x)}{x}\right) \leq \lambda \Lambda\left(\frac{F(x)}{x}\right)$$

$$\leq \lambda \frac{1}{x} \int_0^x \Lambda(f(t))\, dt$$

$$\leq \lambda \frac{1}{x} \int_0^x \Phi^{1/\alpha}(f(t))\, dt .$$

Now we have

$$\int_0^\infty x^s \Phi\left(\frac{F(x)}{x}\right) dx$$

$$\leq \lambda^\alpha \int_0^\infty x^s \left(\frac{1}{x} \int_0^x \Phi^{1/\alpha}(f(t))\, dt \right)^\alpha dx$$

$$\leq A \int_0^\infty x^s \Phi(f(x))\, dx$$

by Lemma 2.

This proves Lemma 3.

6.3 Proof of Theorem 1. Integrating by parts, we have

$$\int_0^x x^s \Phi(G(x))\, dx$$

$$= \left[x^{s+1} \frac{\Phi(G(x))}{s+1} \right]_0^{\pi} - \frac{1}{s+1} \int_0^{\pi} x^{s+1} \phi(G(x)) \left(\frac{f(x)}{x} - \frac{f(x/2)}{x/2} \cdot \frac{1}{2} \right) dx$$

$$\leq \frac{\pi^{s+1} \Phi(G(\pi))}{s+1} + \frac{1}{s+1} \int_0^{\pi} x^{s} \phi(G(x)) \; |f(x) - f(x/2)| \; dx.$$

Taking T to be arbitrary large positive number, we have, since $\phi(x)$ is non-decreasing

$$\phi\,(G(x)) \; |f(x) - f(\tfrac{x}{2})|$$

$$= T^{-1} \left\{ T\,\phi(G(x)) \; |f(x) - f(\tfrac{x}{2})| \right\}$$

$$\leq T^{-1} \operatorname{Max} \{ G(x)\, \phi(G(x)),$$

$$T\,|f(x) - f(\tfrac{x}{2})|\, \phi\,(T\,|f(x) - f(\tfrac{x}{2})|)\,)$$

$$\leq T^{-1}\, \Phi(2\,G(x)) + T^{-1}\, \Phi(2T\,|f(x) - f(\tfrac{x}{2})|\,)$$

$$\leq AT^{-1}\, \Phi(G(x)) + AT^{-1}\, \Phi(\,|f(x) - f(\tfrac{x}{2})|\,)$$

Thus we have

$$\int_0^{\pi} x^{s}\, \Phi(G(x))\, dx$$

$$\leq A\, \Phi(G(\pi)) + \frac{1}{s+1} \int_0^{\pi} x^{s}\, \phi(G(x)) \; |f(x) - f(\tfrac{x}{2})| \; dx$$

$$\leq A\,\Phi\left(\int_{\pi/2}^{\pi} \frac{f(t)}{t}\,dt\right) + \frac{A}{T(s+1)} \int_0^{\pi} x^s\,\Phi(G(x))\,dx$$

$$+ \frac{A}{T(s+1)} \int_0^{\pi} x^s\,\Phi\left(\left|f(x) - f\left(\frac{x}{2}\right)\right|\right) dx$$

or,

$$\left(1 - \frac{A}{T(s+1)}\right) \int_0^{\pi} x^s\,\Phi(G(x))\,dx$$

$$\leq A\,\Phi\left(\frac{2}{\pi}\int_{\pi/2}^{\pi} f(t)\,dt\right) + \frac{A}{T(s+1)} \int_0^{\pi} x^s\,\Phi\left(\left|f(x) - f\left(\frac{x}{2}\right)\right|\right) dx$$

or,

$$\int_0^{\pi} x^s\,\Phi(G(x))\,dx$$

$$\leq A\,\Phi\left(\int_{\pi/2}^{\pi} f(t)\,dt\right) + A\int_0^{\pi} x^s\,\Phi\left(\left|f(x) - f\left(\frac{x}{2}\right)\right|\right) dx$$

by choosing $T > \frac{A}{s+1}$.

This completes the proof of Theorem 1.

6.6 <u>Proof of Theorem 2</u>. First of all we have by Jensen's inequality,

(6.6.1) $\Phi(G(x)) = \left(\int_{\pi/2}^{\pi} \frac{f(t)}{t}\,dt\right)$

$$\leq \Phi\left(\frac{2}{x} \int_{x/2}^{x} f(t)\, dt\right)$$

$$\leq \frac{2}{x} \int_{x/2}^{x} \Phi(f(t))\, dt$$

$$\leq A\, x^{-s-1} \int_{x/2}^{x} t^{s}\, \Phi(f(t))\, dt$$

$$= o(1)\,, \quad x \to 0\,.$$

Now, integrating by parts, we have

$$\int_0^\pi x^s\, \Phi(G(x))\, dx$$

$$= \left[\frac{x^{s+1}\, \Phi(G(x))}{s+1} \right]_0^\pi - \frac{1}{s+1} \int_0^\pi x^{s+1} \phi(G(x)) \left(\frac{f(x)}{x} - \frac{\frac{1}{2} f(\frac{x}{2})}{x/2} \right) dx$$

$$= \frac{\pi^{s+1}\, \Phi(G(\pi))}{s+1} - \frac{1}{s+1} \int_0^\pi x^s\, \phi(G(x)) (f(x) - f(\tfrac{x}{2}))\, dx.$$

First term on the right side is negative since $s < -1$. Hence we have

$$\int_0^\pi x^s\, \Phi(G(x))\, dx$$

$$\leq - \frac{1}{s+1} \int_0^\pi x^s\, \phi(G(x))\, |\, f(x) - f(\tfrac{x}{2})\,|\, dx.$$

Taking T to be arbitrarily large positive integer, we have, since $\phi(x)$ is non-decreasing

$$\phi\,(G(x)\,|\,f(x) - f(\tfrac{x}{2})|)$$

$$= T^{-1}\left\{T\,\phi(G(x))\;\;|f(x) - f(\tfrac{x}{2})|\right\}$$

$$\leq T^{-1}\,\mathrm{Max}\left\{G(x)\;\phi(G(x)),\right.$$

$$\left. T\,|f(x) - f(\tfrac{x}{2})|\;\phi\,(T\,|f(x) - f(\tfrac{x}{2})|\,)\right\}$$

$$\leq T^{-1}\,\Phi\,(2(G(x)) + T^{-1}\,\Phi\,(T\,2|f(x) - f(\tfrac{x}{2})|\,)$$

$$\leq AT^{-1}\,\Phi\,(G(x) + AT^{-1}\,\Phi\,(\,|f(x) - f(\tfrac{x}{2})|\,).$$

This we have

$$\int_0^{\pi} x^s\,\Phi(G(x))\;dx$$

$$\leq \frac{-1}{s+1}\int_0^{\pi} x^s\,\phi(G(x))\;|\,f(x) - f(\tfrac{x}{2})|\;dx$$

$$\leq \frac{-A}{(s+1)T}\int_0^{\pi} x^s\,\Phi\,(G(x))\,dx - \frac{A}{(s+1)\,T}\int_0^{\pi} x^s\,\Phi\,(\,|f(x) - f(\tfrac{x}{2})|\,)\,dx$$

or,

$$\left(1 + \frac{A}{(s+1)T}\right) \int_0^x x^s \; \Phi(G(x)) \; dx$$

$$\leq - \frac{A}{(s+1)T} \int_0^x x^s \; \Phi\left(\left|f(x) - f\left(\frac{x}{2}\right)\right| \right) dx$$

or,

$$\int_0^x x^s \, \Phi(G(x)) \; dx \quad < \quad A \int_0^x x^s \, \Phi\left(\left|f(x) - f\left(\frac{x}{2}\right)\right|\right) \; dx$$

by choosing $T > - \frac{A}{s+1}$.

This completes the proof of Theorem 2.

6.7 Proof of Theorem 3. It is easy to verify that the relation (6.6.1) holds also when $x \to \infty$ and then following the lines of proof of Theorem 2 we have

$$\int_0^\infty x^s \; \Phi(G(x)) \; dx$$

$$= - \frac{\lambda}{s+1} \int_0^\infty x^s \, \phi\,(G(x)) \; \left(f(x) - f\left(\frac{x}{2}\right) \right) dx$$

$$\leq \frac{\lambda}{|s+1|} \int_0^\infty x^s \, \phi\,(G(x)) \; \left|f(x) - f\left(\frac{x}{2}\right)\right| dx$$

$$\leq \frac{A}{T\,|s+1|} \int_0^\infty x^s \, \Phi\,(G(x)) \; dx \; +$$

$$+ \frac{A}{T\,|s+1|} \int_0^\infty x^s \, \Phi\left(\left|f(x) - f\left(\frac{x}{2}\right)\right| \right) dx$$

or,

$$\int_0^\infty x^s \Phi(G(x))\ dx \leq \frac{A}{T.|s+1|} \int_0^\infty x^s \ \Phi(\ |f(x) - f(\tfrac{x}{2})|\ )dx.$$

This proves Theorem 3.

6.8 <u>Proof of Theorem</u> 4. Let γ_n be the Fourier coefficients of $Q(x)$ of Lemma 1. Then by virtue of this lemma, $\gamma_n \geq 0$ for all $n \geq 1$. We put

$$Q_1(x) = \int_0^x Q(t)\ dt\ ,$$

$$Q_2(x) = \int_0^x Q_1(t)\ dt \qquad \text{for } x \geq 0\ .$$

Then

$$Q_2(x) = \frac{\gamma_0\, x^2}{4} + \sum_{j=1}^{\infty} \gamma_j\ j^{-2}\ (1 - \cos jx)$$

$$\geq \sum_{j=1}^{\left[\frac{n}{2}\right]} \gamma_j .\ j^{-2}\ (1 - \cos jx)$$

$$= 2 \sum_{j=1}^{\left[\frac{n}{2}\right]} \gamma_j\ \ j^{-2}\ \ \sin^2 \frac{jx}{2}$$

$$\geq 2 \sum_{j=1}^{\left[\frac{n}{2}\right]} \gamma_j .\ j^{-2}\ \frac{4}{\pi^2} .\ \frac{j^2\, x^2}{4}\ ,\quad \frac{\pi}{n+1} \leq x \leq \frac{\pi}{n}\ .$$

$$= \frac{\varrho}{x^{\varrho}}\; x^{\varrho} \sum_{j=1}^{\left[\frac{n}{\varrho}\right]} \gamma_j$$

$$\geq \frac{\varrho}{\pi^{\varrho}}\, \theta\, x^{\varrho} \sum_{j=1}^{\left[\frac{n}{\varrho}\right]} \frac{1}{j} \sideset{}{^*}\sum_{k=j}^{\varrho j} a_k\,, \quad 0 < \theta < 1\,,$$

$$\geq A\; x^{\varrho} \sum_{j=1}^{\left[\frac{n}{\varrho}\right]} a_{\varrho j}$$

$$\geq A\; x^{\varrho}\, n\, a_n\,,$$

or, we have

$$n\, a_n \leq \frac{1}{A}\, \frac{Q_{\varrho}(x)}{x^{\varrho}}, \quad \frac{\pi}{n+1} \leq x \leq \frac{\pi}{n}\,, \quad n \geq 1.$$

Now, we have

$$\sum_{n=1}^{\infty} n^{-s-\varrho}\, \bar{\Phi}(\, n\, a_n\,)$$

$$\leq A \sum_{n=1}^{\infty} \int_{\pi/_{n+1}}^{\pi/n} x^{s}\; \bar{\Phi}\left(\frac{Q_{\varrho}(x)}{A\, x^{\varrho}}\right) dx$$

$$\leq A \int_0^{\pi} x^{s}\; \bar{\Phi}\left(\frac{Q_{\varrho}(x)}{x^{\varrho}}\right) dx$$

$$= A \int_0^{\pi} x^{s}\; \bar{\Phi}\left(\frac{1}{x^{\varrho}} \int_0^{x} Q_1(t)\; dt\,\right) dx$$

$$\leq A \int_0^\pi x^s \, \Phi\left(\frac{G_1(x)}{x}\right) dx$$

$$\leq A \int_0^\pi x^s \, \Phi(G(x)) \, dx$$

by Lemma 3.

Now, by Theorem 1, we have

$$\sum_{n=1}^{\infty} n^{-s-2} \Phi(n\, a_n)$$

$$\leq A \int_0^\pi x^s \, \Phi(G(x)) \, dx$$

$$\leq A \int_0^\pi x^s \, \Phi\left(\left| f(x) - f\left(\frac{x}{2}\right) \right| \right) dx$$

$$+ A \, \Phi\left(\int_{\pi/2}^{\pi} f(x) \, dx \right).$$

This proves part (i) of Theorem 4.

Now, if we suppose that $s < -1$ and use Theorem 2, then we get

$$\sum_{n=1}^{\infty} n^{-s-2} \, \Phi(n\, a_n) \leq A \int_0^\pi x^s \, \Phi(G(x)) \, dx$$

$$\leq A \int_0^\pi x^s \, \Phi\left(\left| f(x) - f\left(\frac{x}{2}\right) \right| \right) dx.$$

This proves part (ii) of Theorem 4.

Chapter VII

INTEGRABILITY THEOREMS FOR FOURIER SERIES AND PARSEVALS FORMULAE

7.1 Recently, Hasegawa [1] has proved, among others, the following theorems.

Theorem A. Let $f(x)$ be an even function, continuous on $(0,\pi)$, and let its Fourier series be

$$(7.1.1) \qquad f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx.$$

Suppose that $\alpha(x)$ is an even function, positive and non-increasing on $(0,\pi)$, that $x\alpha(x) \in L(0,\pi)$ and that there is a positive number $\eta \leq \pi$ such that

$$t^{-2} \int_0^t x\,\alpha(x)\,dx \leq C^{\dagger} \alpha(t) \text{ for all } t, \quad 0 < t \leq \eta.$$

If the series $\sum_{n=1}^{\infty} a_n \int_{1/n}^{\pi} \alpha(x)\,dx$ converges absolutely, then the Fourier series (7.1.1) converges uniformly to $f(x)$ and

† where C, with or without suffixes, denotes a positive constant, not necessarily the same at each occurrence.

$(f(x) - f(s))\, \alpha(x-s) \in L(0,\pi)$ for each s, $0 < s < \pi$.

<u>Theorem B</u>. Let $g(x)$ be an odd function, continuous on $(0,\pi)$, and let its Fourier series be

$$(7.1.2) \qquad g(x) \sim \sum_{n=1}^{\infty} b_n \sin nx.$$

Suppose that $\alpha(x)$ is an odd function, positive on $(0,\pi)$, that $x\,\alpha(x)$ is Lebesgue integrable and non-increasing on $(0,\pi)$ and that there is a positive number $\eta \leq \pi$ such that

$$t^{-1} \int_t^{\eta} \alpha(x)\, dx \leq C\, \alpha(t) \qquad \text{for all } t,\ 0 < t \leq \eta.$$

If the series $\sum_{n=1}^{\infty} n\, b_n \int_0^{1/n} x\,\alpha(x)\, dx$ converges absolutely, then the Fourier series (7.1.2) converges uniformly to $g(x)$ and $(g(x)-g(s))\, \alpha(x-s) \in L(0,\pi)$ for each s, $0 \leq s \leq \pi$. In particular, in case $s=0$, Parseval's formulae

$$\frac{2}{\pi} \int_0^{\pi} g(x)\, \alpha(x)\, dx = \sum_{n=1}^{\infty} b_n q_n$$

holds, where q_n's are defined by

$$q_n = \frac{2}{\pi} \int_0^{\pi} \alpha(x) \sin nx\, dx\ ,$$

and the series $\sum_{n=1}^{\infty} b_n q_n$ converges absolutely.

The object of this chapter is to obtain certain generalisation of these theorems. In what follows we assume that $\alpha(x)$, $\phi(x)$ and $\psi(x)$ are positive functions defined on $(0,\pi)$ such that $\alpha(x)\,\phi(x) \in L(0,\pi)$ and $\alpha(x)\,\psi(x)$ is either even or odd.

7.2 We prove the following theorems.

Theorem 1. Let $f(x)$ be an even function and continuous on $(0,\pi)$ and let its Fourier series be (7.1.1). Suppose that $\alpha(x)\dfrac{\psi(x)}{\phi(x)}$ is non-increasing on $(0,\pi)$ and that there is a positive number $\eta \leq \pi$ such that

$$\text{(7.2.1)} \qquad \int_0^t \phi(x)\,\alpha(x)\,dx \leq C\,\Phi(t)\,\alpha(t)$$

for all t, $0 < t \leq \eta$, where $\Phi(t) = \int_0^t \phi(x)\,dx$. If the series

$$\sum_{n=1}^{\infty} a_n \int_{1/n}^{\pi} \alpha(x)\,\psi(x)\,dx$$

converges absolutely, then the Fourier series (7.1.1) converges uniformly to $f(x)$ and

$$(\,f(x) - f(s)\,)\,\alpha\,(x-s)\,\psi(x-s) \in L(0,\pi)$$

for each s, $0 < s < \pi$, where $x\,\dfrac{\psi(x)}{\phi(x)} < M^{*}$, $0 < x \leq \pi$

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* where M is a positive constant.

and

$$(7.2.2) \qquad \frac{\phi(x)\ \Phi(x)}{\psi(x)\,(\Phi(x) - \Phi(x/2))} = O(x), \quad x \to 0.$$

Theorem 2. Let $g(x)$ be an odd function, continuous on $(0,\pi)$ and let its Fourier series be (7.1.2). Suppose that $\phi(x)\ \alpha(x)$ is non-increasing on $(0,\pi)$ and that there is a positive number $\eta \leq \pi$ such that

$$(7.2.3) \qquad \int_t^{\eta} \alpha(x)\psi(x)\ dx \leq C\ \phi(t)\ \alpha(t)$$

for all t, $0 < t \leq \eta$, where

$$x\,\frac{\psi(x)}{\phi(x)} < M, \qquad 0 < x \leq \pi .$$

If the series

$$\sum_{n=1}^{\infty} n\ b_n \int_0^{1/n} \phi(x)\ \alpha(x)\ dx$$

converges absolutely, then the Fourier series (7.1.2) converges uniformly to $g(x)$ and $(g(x) - g(s))\ \alpha(x-s)\ \psi(x-s) \in L(0,\pi)$ for each s, $0 \leq s \leq \pi$. In particular for $s=0$, Parseval's formulae

$$\frac{2}{\pi}\int_0^{\pi} g(x)\ \alpha(x)\ \psi(x)\ dx = \sum_{n=1}^{\infty} b_n\ q_n$$

<u>holds and the series</u> $\sum_{n=1}^{\infty} b_n q_n$ <u>converges absolutely, where</u>

$$(7.2.4) \qquad q_n = \frac{2}{\pi}\int_0^{\pi} \alpha(x)\ \psi(x)\ \sin nx\ dx.$$

<u>Remarks</u>. By putting $\psi(x) = 1$ and $\phi(x) = x$ in Theorems 1 and 2, we get Theorems A and B respectively. On the other hand, if we take $\phi(x) = x$ and write $\psi(x)\ \alpha(x) = \beta(x)$ in Theorem 1, we get Theorem A in which one of our conditions namely " $\frac{\beta(x)}{x}\downarrow$ " is lighter than the corresponding condition "$\beta(x)\downarrow$ " of Theorem A. By taking $\psi(x) = 1+x$, and $\phi(x) = x$ a similar remark is applicable to Theorem B. Also we extend the scope of our theorems by assuming that $\beta(x)$ is either even or odd instead of restricting it to be even in Theorem A and odd in Theorem B. Other interesting cases for which our theorems hold are

(i) $\quad \psi(x) = \sin\frac{x}{2}\ , \quad \phi(x) = x^2$.

(ii) $\quad \psi(x) = \cos\frac{x}{2}\ , \quad \phi(x) = x.$

7.3 <u>Proof of Theorem 1</u>. Since $\sum_{n=1}^{\infty} a_n \int_{1/n}^{\pi} \alpha(x)\ \psi(x)\ dx$ converges absolutely, we have

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left\{ |a_n| \frac{\int_{1/n}^{\pi} \alpha(x)\ \psi(x)\ dx}{\int_{1/n}^{\pi} \alpha(x)\ \psi(x) dx} \right\}$$

$$\leq \left(\int_1^\pi \alpha(x)\, \psi(x)\,dx \right)^{-1} \sum_{n=1}^{\infty} |a_n| \int_{1/n}^{\pi} \alpha(x)\,\psi(x)\,dx$$

$$< \infty .$$

By hypothesis $f(x)$ is continuous and hence in view of well known result* the Fourier series (7.1.1) converges uniformly to $f(x)$.

Since $\dfrac{\alpha(x)\,\psi(x)}{\phi(x)}$ is non-increasing, we have

$$\int_{1/n}^{2/n} \alpha(x)\,\psi(x)\,dx$$

$$\geq \frac{\alpha(2/n)\,\psi(2/n)}{\phi(2/n)} \int_{1/n}^{2/n} \phi(x)\,dx$$

$$= \frac{\alpha(2/n)\,\psi(2/n)}{\phi(2/n)} \left(\Phi(2/n) - \Phi(1/n) \right),$$

or,

$$\alpha(2/n) \leq \frac{\phi(2/n)}{\psi(\frac{2}{n}) \left(\Phi(\frac{2}{n}) - \Phi(\frac{1}{n}) \right)} \int_{1/n}^{2/n} \alpha(x)\,\psi(x)\,dx$$

$$\leq \frac{\phi(2/n)}{\psi(\frac{2}{n}) \left(\Phi(\frac{2}{n}) - \Phi(\frac{1}{n}) \right)} \int_{1/n}^{\pi} \alpha(x)\,\psi(x)\,dx.$$

Hence from (7.2.1) and (7.2.2) we have

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*see for example Zygmund [1].

$$(7.3.1) \qquad \int_0^{1/n} \phi(x)\alpha(x)\,dx \le \int_0^{2/n} \phi(x)\alpha(x)\,dx$$

$$\le C\,\alpha\,(2/n)\,\Phi\,(2/n)$$

$$\le C\,\frac{\phi(\frac{2}{n})\;\Phi(\frac{2}{n})}{\Psi(\frac{2}{n})\left(\Phi(\frac{2}{n})-\Phi(\frac{1}{n})\right)} \cdot \int_{1/n}^{\pi} \alpha(x)\;\Psi(x)\,dx$$

$$\le \frac{C_1}{n}\int_{1/n}^{\pi} \alpha(x)\;\Psi(x)\,dx$$

for all $n \ge N$, where N denotes the smallest integer $\ge \frac{1}{\eta}$. Since $\phi(x)\;\alpha(x) \in L(0,\pi)$ and $\Psi(x)\;\alpha(x)$ is either even or odd, we have for $0 < s < \pi$

$$\int_0^{\pi} \left|(f(x)-f(s)\;)\;\alpha(x-s)\;\Psi(x-s)\right|\;dx$$

$$= \int_0^{\pi} \left|\left(\sum_{n=1}^{\infty} a_n(\cos nx - \cos ns)\right)\alpha(x-s)\;\Psi(x-s)\right|\;dx$$

$$\le \sum_{n=1}^{\infty} |a_n| \int_0^{\pi} \left|(\cos nx - \cos ns)\;\alpha(x-s)\;\Psi(x-s)\right|\;dx$$

$$\le 2\sum_{n=1}^{\infty} |a_n| \int_0^{\pi} \left|\sin \tfrac{n}{2}\,(x-s)\;\alpha(x-s)\;\Psi(x-s)\right|\;dx$$

$$= 2\sum_{n=1}^{\infty} |a_n| \int_{-s}^{\pi-s} \left|\sin \tfrac{nx}{2}\,\alpha(x)\;\Psi(x)\right|\;dx$$

$$= 2 \sum_{n=1}^{\infty} |a_n| \left(\int_{-\varepsilon}^{0} + \int_{0}^{\pi-\varepsilon} \right) \left|\sin\frac{nx}{2}\, \alpha(x)\ \psi(x)\right| \, dx$$

$$\leq 2 \sum_{n=1}^{\infty} |a_n| \left(\int_{0}^{\varepsilon} + \int_{0}^{\pi-\varepsilon} \right) \left|\sin\frac{nx}{2}\right| \alpha(x)\ \psi(x) \, dx$$

$$\leq 4 \sum_{n=1}^{\infty} |a_n| \int_{0}^{\pi} \left|\sin\frac{nx}{2}\right| \alpha(x)\ \psi(x)\, dx$$

$$= 4 \sum_{n=1}^{N} |a_n| \int_{0}^{\pi} \left|\sin\frac{nx}{2}\right| \alpha(x)\ \psi(x)\, dx$$

$$+ 4 \sum_{n=N+1}^{\infty} |a_n| \left(\int_{0}^{1/n} + \int_{1/n}^{\pi} \right) \left|\sin\frac{nx}{2}\right| \alpha(x)\ \psi(x)\, dx$$

$$= I_1 + I_2, \text{ say.}$$

Now

$$I_2 \leq 2 \sum_{n=N+1}^{\infty} n\, |a_n| \int_{0}^{1/n} x\, \alpha(x)\ \psi(x)\, dx +$$

$$+ 4 \sum_{n=N+1}^{\infty} |a_n| \int_{1/n}^{\pi} \alpha(x)\ \psi(x)\, dx$$

$$\leq C \sum_{n=N+1}^{\infty} n\, |a_n| \int_{0}^{1/n} \alpha(x)\phi(x)\, dx + 4 \sum_{n=N+1}^{\infty} |a_n| \int_{1/n}^{\pi} \alpha(x)\ \psi(x)\, dx$$

$$\leq C_1 \sum_{n=N+1}^{\infty} |a_n| \int_{1/n}^{\pi} \alpha(x)\ \psi(x)\, dx$$

$$< \infty,$$

by (7.3.1) and the hypothesis.

Also,

$$I_1 \leq 4 \left(\int_0^{\pi} x\, \psi(x)\, \alpha(x)\, dx\right) \sum_{n=1}^{N} n\, |a_n|$$

$$\leq C \left(\int_0^{\pi} \alpha(x)\, \phi(x)\, dx \right) \sum_{n=1}^{N} n\, |a_n|$$

$$< \infty .$$

Hence $|f(x) - f(s)|\, \alpha(x-s)\, \psi(x-s) \in L\,(0,\pi)$.

Thus Theorem 1 is established.

7.4 <u>Proof of Theorem 2</u>. Since $\phi(x)\, \alpha(x)$ is positive, non-increasing and Lebesgue integrable on $(0,\pi)$, we have

$$\int_0^{1/n} \phi(x)\, \alpha(x)\, dx \geq \frac{1}{n}\, \phi\!\left(\frac{1}{n}\right) \alpha\!\left(\frac{1}{n}\right) \geq \frac{1}{n}\, \phi(1)\, \alpha(1).$$

According to the hypothesis, $\sum_{n=1}^{\infty} n\, b_n \int_0^{1/n} \phi(x)\, \alpha(x)\, dx$ converges absolutely and hence we have

$$\sum_{n=1}^{\infty} |b_n| = \sum_{n=1}^{\infty} |b_n| \frac{n \int_0^{1/n} \phi(x)\, \alpha(x)\, dx}{n \int_0^{1/n} \phi(x)\, \alpha(x)\, dx}$$

$$\leq \frac{1}{\phi(1)\alpha(1)} \sum_{n=1}^{\infty} n\, |b_n| \int_0^{1/n} \alpha(x)\, \phi(x)\, dx$$

$$< \infty .$$

consequently, as in the proof of Theorem 1, the Fourier series (7.1.2) converges uniformly to $g(x)$.

Now using condition (7.2.3)

$$(7.4.1) \qquad \int_{1/n}^{\eta} \alpha(x)\,\psi(x)\,dx \le C\,\phi\left(\frac{1}{n}\right)\alpha\left(\frac{1}{n}\right)$$

$$\le C\,n \int_0^{1/n} \alpha(x)\,\emptyset(x)\,dx$$

for all $n \ge N$, where N denotes the smallest integer $\ge \frac{1}{\eta}$. We first assume that $0 < s < \pi$. Then

$$\int_0^{\pi} \left|(g(x) - g(s))\,\alpha(x-s)\,\psi(x-s)\right| dx$$

$$= \int_0^{\pi} \left|\left(\sum_{n=1}^{\infty} b_n(\sin nx - \sin ns)\right)\alpha(x-s)\,\psi(x-s)\right| dx$$

$$\le 2 \sum_{n=1}^{\infty} |b_n| \int_0^{\pi} \left|\sin \frac{n(x-s)}{2}\,\alpha(x-s)\,\psi(x-s)\right| dx$$

$$= 2 \sum_{n=1}^{\infty} |b_n| \int_{-s}^{\pi-s} \left|\sin \frac{nx}{2}\,\alpha(x)\,\psi(x)\right| dx$$

$$= 2 \sum_{n=1}^{\infty} |b_n| \left(\int_{-s}^{0} + \int_0^{\pi-s}\right) \left|\sin \frac{nx}{2}\,\alpha(x)\,\psi(x)\right| dx$$

$$\le 2 \sum_{n=1}^{\infty} |b_n| \left(\int_0^{s} + \int_0^{\pi-s}\right) \left|\sin \frac{nx}{2}\right| \alpha(x)\,\psi(x)\,dx$$

$$(7.4.2) \qquad \le 4 \sum_{n=1}^{\infty} |b_n| \int_0^{\pi} \left|\sin \frac{nx}{2}\right| \alpha(x)\,\psi(x)\,dx$$

$$= 4 \left(\sum_{n=1}^{N} + \sum_{n=N+1}^{\infty} \right) |b_n| \int_0^{\pi} \left|\sin \frac{nx}{2}\right| \alpha(x)\ \psi(x)\ dx$$

$$= J_1 + J_2 , \quad \text{say.}$$

Now,

$$J_2 = 4 \sum_{n=N+1}^{\infty} |b_n| \left(\int_0^{1/n} + \int_{1/n}^{\eta} + \int_{\eta}^{\pi} \right) \left|\sin \frac{nx}{2}\right| \alpha(x)\ \psi(x)\ dx$$

$$\leq 2 \sum_{n=N+1}^{\infty} n\, |b_n| \int_0^{1/n} x\, \alpha(x)\ \psi(x)\ dx \ +$$

$$+ \ 4 \sum_{n=N+1}^{\infty} |b_n| \int_{1/n}^{\eta} \alpha(x)\ \psi(x)\ dx \ +$$

$$+ \ 4 \sum_{n=N+1}^{\infty} |b_n| \int_{\eta}^{\pi} \alpha(x)\ \psi(x)\ dx$$

$$\leq C \sum_{n=N+1}^{\infty} n\, |b_n| \int_0^{1/n} \alpha(x)\ \phi(x)\ dx$$

$$+ \ C_1 \sum_{n=N+1}^{\infty} n\, |b_n| \int_0^{1/n} \alpha(x)\ \phi(x)\, dx + C_2$$

$$< \infty ,$$

by virtue of (7.4.1) and the hypothesis.

Also,

$$J_1 \leq 2 \sum_{n=1}^{N} n\, |b_n| \int_0^{\pi} x\, \alpha(x)\ \psi(x)\ dx$$

$$\leq C \left(\sum_{n=1}^{N} n \, |b_n| \right) \int_0^{\pi} \alpha(x) \, \phi(x) \, dx$$

$$< \infty .$$

Hence $(g(x) - g(s)) \, \alpha(x-s) \, \psi(x-s) \in L(0,\pi)$ for $0 < s < \pi$.

The cases $s=0$ and $s=\pi$ can be easily disposed off in a similar manner.

To prove the last part of our assertion we notice that

$$(7.4.3) \qquad \sum_{n=1}^{\infty} |b_n \, q_n|$$

$$\leq \sum_{n=1}^{\infty} |b_n| \int_0^{\pi} \alpha(x) \, \psi(x) \, |\sin nx| \, dx$$

$$< \infty ,$$

as shown in (7.4.2).

Thus $\sum_{n=1}^{\infty} b_n \, q_n$ converges absolutely and hence

$$\sum_{n=1}^{\infty} b_n \, q_n = \sum_{n=1}^{\infty} b_n . \frac{2}{\pi} \int_0^{\pi} \alpha(x) \, \psi(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \alpha(x) \, \psi(x) \left(\sum_{n=1}^{\infty} b_n \sin nx \right) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \alpha(x) \, \psi(x) \, g(x).$$

The term by term integration is justified in view of (7.4.3).

This completes the proof of Theorem 2.

Chapter VIII

THE INTEGRABILITY CLASS OF THE SINE TRANSFORM OF A MONOTONIC FUNCTION

8.1 A non-decreasing, continuous and real-valued function Φ defined on the non-negative half-line and vanishing only at the origin is called an Orlicz Function (OF). Function $\Phi \in$ OF is said to satisfy Δ_2 - condition for large u if there are constants $c > 0$ and $u_0 \geq 0$ such that

$$\Phi(2u) \leq c\ \Phi(u), \quad u \geq u_0.$$

8.2 Recently, Boas [6] has established the following theorem for Fourier transform by a method which is rather more direct than those that have been used for similar problems about Fourier series. His method depends on the Steffensen's version of Jensen's inequality (see Mitrinović [1] , p.109) and a theorem of Edmonds [1] on Parseval's theorem for monotonic function.

__Theorem A__. If $f(x) \downarrow 0$, $x^{1/p} f(x) \in L^p(0,1)$, and

$$F(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin xt \, dt ,$$

then $x^{\gamma+1-\frac{2}{p}} F(x) \in L^p (0, \infty)$ provided that

$$x^{-\gamma} f(x) \in L^p (0, \infty) ,$$

where $p > 1$ and $-\frac{1}{p} < \gamma < \frac{1}{p}$.

It may be remarked that in Theorem A, the condition $x^{1/p} f(x) \in L^p (0, 1)$ need not be mentioned because it is already implied by the condition

$$x^{-\gamma} f(x) \in L^p (0, \infty), \ -\frac{1}{p} < \gamma < \frac{1}{p} .$$

In this chapter it is proposed to obtain a generalization of the above theorem. Instead of considering L^p class we would employ a more general class, namely L_{Φ} .

8.3 We prove the following theorem.

<u>Theorem</u>. <u>Let</u> $F(x)$ <u>be the sine transform of</u> $f(x)$. <u>If</u> $f(x) \downarrow 0$, $x^{-\alpha} \Phi (f(x)) \in L (0, \infty)$ <u>and</u> $-1 < \alpha < 1$, <u>then</u> $\left(x^{\alpha-2}\right) \Phi (x F(x)) \in L (0, \infty)$, <u>where</u> $\Phi (x)$ <u>is a</u>

convex Orlicz function satisfying Δ_2 condition.

It may be observed that for $\Phi(t) = t^p$, $p > 1$ we get theorem A.

8.4 We require the following lemmas for the proof of our theorem.

Lemma 1. (Boas [8]). Let λ be a function of bounded variation on every finite sub-interval of $(0, \infty)$; $\lambda(o) \leq \lambda(x)$ for all $x > 0$; and $\lambda(o) < \Lambda = \sup \lambda(x)$. Let $f(x)$ decrease and $f(x) \geq 0$. If ψ is continuous and convex over $(0, f(o))$, $\psi(o) \leq 0$ and

$$\int_0^\infty d\,\mu(x) \geq \Lambda - \lambda(o) ,$$

then

$$\psi\left\{\frac{\int_0^\infty f(x)\ d\lambda(x)}{\int_0^\infty d\ u(x)}\right\} \leq \frac{\int_0^\infty \psi(\ f(x)\ )\ d\lambda(x)}{\int_0^\infty d\,\mu(x)}$$

Lemma 2. (Boas [3] , p. 58). If g and B decrease to 0 on $(0, \infty)$ and $x\,g(x)$, $x\,B(x) \in L(0,1)$, then $B(y)\ b(y) \in L\ (0, \infty)$ if, and only if , $g(u)\ G(u) \in L\ (0, \infty)$ and Parseval's formula

$$\int_0^\infty B(y)\ b(y)\ dy = \int_0^\infty G(u)\ g(u)\ du$$

holds, G and b being the sine transforms of B and g respectively.

8.5 Proof of the Theorem. Taking $\lambda(x) = 1-\cos x$, $\Lambda = 2$ and using Lemma 1, we have

(8.5.1)
$$\Phi\left(\frac{1}{2}\int_0^\infty f(x)\ \sin x\ dx\right) \leq \frac{1}{2}\int_0^\infty \Phi(f(x))\ \sin x\ dx\ .$$

Since sine transform of a positive decreasing function is positive, it follows that reight-hand side is positive. Also in view of the hypotheses, it is finite. Now replacing $f(x)$ by $f(xt)$, multiplying (8.5.1) by $t^{-\alpha}$ and integrating over $(0, \infty)$ we have

(8.5.2)
$$\int_0^\infty t^{-\alpha}\ \Phi\left(\frac{1}{2}\int_0^\infty f(xt)\ \sin x\ dx\right)\ dt \leq \frac{1}{2}\int_0^\infty t^{-\alpha}\ dt\ \int_0^\infty \Phi(f(xt))\ \sin x\ dx.$$

Putting $t = \frac{1}{y}$ and $x = yu$ in (8.5.2) we have

$$\int_0^\infty y^{\alpha-2}\ \Phi\left(\frac{1}{2}\ f(u)\ \sin yu\ ydu\right)\ dy$$

$$\leq \frac{1}{2} \int_0^\infty y^{\alpha-2} dy \int_0^\infty \Phi(f(u)) \sin yu \, y du.$$

That is to say,

$$\int_0^\infty y^{\alpha-2} \Phi(\frac{1}{2} \sqrt{\frac{\pi}{2}} y F(y)) dy$$

$$\leq \frac{1}{2} \int_0^\infty y^{\alpha-1} dy \int_0^\infty \Phi(f(u)) \sin yu \, du .$$

Thus, it follows that,

$$\int_0^\infty y^{\alpha-2} \Phi(y F(y)) dy$$

$$\leq C \int_0^\infty y^{\alpha-2} \Phi(\frac{1}{2} \sqrt{\frac{\pi}{2}} y F(y)) dy$$

$$\leq C \int_0^\infty y^{\alpha-1} dy \int_0^\infty \Phi(f(u)) \sin yu \, du$$

$$= C \int_0^\infty B(y) b(y) dy ,$$

where $B(y) = y^{\alpha-1}$, $-1 < \alpha < 1$, $g(u) = \Phi(f(u))$ and G and b are the sine transforms of B and g respectively.

Now

$$\int_0^\infty g(u) G(u) du$$

$$= \int_0^\infty \Phi(f(u)\ du \int_0^\infty y^{\alpha-1} \sin yu\ du$$

$$= \int_0^\infty u^{-\alpha} \Phi(f(u))\ du \int_0^\infty t^{\alpha-1} \sin t\ dt\ ^{*}$$

$$= \Gamma(\alpha) \sin\frac{\pi\alpha}{2} \int_0^\infty u^{-\alpha} \Phi(f(u))\ du$$

$$< \infty ,$$

by virtue of the hypotheses. Thus $Gg \in L(0, \infty)$. In view of Lemma 2 , Parseval's formula holds and therefore,

$$\int_0^\infty y^{\alpha-2} \Phi(y F(y))\ dy < \infty .$$

Thus our theorem is proved.

---

* when $\alpha = 0$, the integral $\int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}$.

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